Analysis of the Nash Demand Game

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(very rough draft)

1 Description of the Game

Described by John Nash in 1953, the game is a simple bargaining situation in which two agents make simultaneous "take-it-or-leave-it" bids. The bids take the form of fractions of a good to be divided between them. If the sum of the bids exceeds one the payoff is zero for both agents. Otherwise, each agent receives his bid.

The game can be considered a model for economic interactions in which the cost of negotiation is prohibitively high. In H. Peyton Young's *The Economics of Convention*, the game is used to model situations in which each agent relies on convention, the historical behavior of agents in his situation, to determine his opponent's action. Negotiation can be neglected in the case of conventions, because there are no external factors (i.e. moral desert) to apply to the case.

Young proposes the convention of driving on a particular side of the road as an example. In this case the worth of driving on the right or left side depends entirely on the other drivers; there is no inherent value to driving on either side. One's behavior is therefore likely to follow a convention, which makes negotiation unnecessary.

Another example is that of the sharecropping system. The fraction of the yield a tenant must give to the landlord is arbitrary, assuming that it is within a range that makes the system worthwhile for both parties. When dividing up the surplus, negotiation is unnecessary insofar as the outcome is Pareto efficient. More generally, a buyer and seller agree on price that generally reflects the relative cost of no sale being made to either agent (cite? or demonstrate?). Each wants as much as he can get, but when he expects the other not to budge from a particular price, he is left with no choice but to accept that price. Many bargaining situations with a surplus to be divided between two agents fit this paradigm.

Hence the Nash demand game is useful for situations in which either negotiation is costly or negotiation is rendered useless by the lack of any principle to apply, other than each agent trying to get as much as he can.

2 The Game Played Within a Population

We model the Nash demand game as occurring sequentially between random pairs within a population of N agents. Each agent calculates his bid from his memory of the last m bids played against him, and assumes his memory to be representative of the play of the next agent he meets. We start by investigating the dynamics of the bids in the framework of discrete bidding levels. If the number of levels is D equally spaced bids within the unit interval, the allowed bids are

$$\left\{ \frac{1}{D+1}, \frac{2}{D+1}, \dots, \frac{D}{D+1} \right\}$$

The final parameter in this model is the noise ϵ, which is the fraction of bids that are randomly drawn from a uniform distribution over the possible bid levels. The effect of the noise is to allow the population to move to the stochastically stable equilibrium of nearly all agents bidding 0.5, apart from random bids and the degree to which they perturb the calculated bids.

3 Example: D=3

Now we examine the system in the case of three bid levels: 0.25, 0.5, and 0.75. Consider an agent with memory length m=5. His memory is currently

$$\{0.25, 0.75, 0.50, 0.50, 0.25\}$$

To maximize the expected value of his next bid, he assumes his opponent will play each of the three bids with probability equal to the proportion they occupy in his memory. Thus

$$EV(0.25) = 0.25 \cdot \frac{5}{5} = 0.25 \tag{1}$$

$$EV(0.50) = 0.50 \cdot \frac{4}{5} = 0.40 \tag{2}$$

$$EV(0.75) = 0.75 \cdot \frac{2}{5} = 0.30 \tag{3}$$

Thus with probability $1-\epsilon$ he bids 0.50 and with probability ϵ he bids randomly.

If $\epsilon > 0$, the system will move within a finite amount of time from any initialization to $1 - \frac{2\epsilon}{3}$ agents bidding 0.50, on average (Fig. **??**).

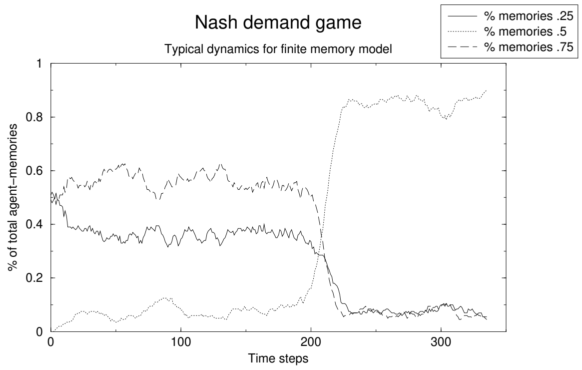

Figure 1: Fraction of Agents' Memories Occupied by Each Bid Level, $D = 3$.

The amount of time it takes for the ensemble to reach 50% of agents playing 0.50, roughly the midpoint of the transition in Figure **??**, is defined as the "waiting time", T. There are essentially two classes of behavior for T, each corresponding to a different type of memory initialization. The first, with which we are concerned, is when there are no 0.50 in the agents' memories at the start. Then, in the absence of noise, the agents' memories will all contain 0.75 and 0.25 in a 2:1 ratio. This is the only possible equilibrium with only 0.75 and 0.25 in memory. From this point, there will be no significant change in the memory content until, purely from random bids, the number of 0.50 bids in some agents' memories exceeds $\frac{1}{6}$. These agents will begin to bid 0.50 systematically, causing more agents' memories to contain 0.50, and soon the whole population shifts as in Figure **??**. It turns out that the number of such agents required to shift the whole population is also $\frac{1}{6}$ (explain!!). The second type of initialization, in which there are 0.50 in the agents' memories, will simply exhibit shorter waiting times than the first type initialization.

(explain theory behind double exponential here) The behavior of the waiting time as a function of m is plotted for $N = 10$ and $N = 100$ (Fig. **??**).

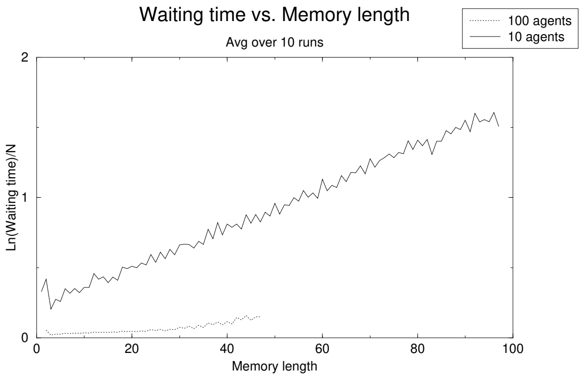

Figure 2: Waiting Time as a Function of Memory Length, $N = 10,\ N = 100$.

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3.1 Periodicity in Waiting Time

Explain the period six behavior in waiting time, and note that for 100 agents the minimum T for each set of six occurs for memories of the form $3k + 6$. For $N = 10$, the big jumps occur at: 6,12,18,24,30,36,44,50,56,60,64?,66?,70,78,86,90?,92?. So the periodicity need be examined more carefully here. The dips are also hard to locate (though they're always odd. So maybe just run odd mem_lengths?)

3.2 The Effect of Noise and Discretization

Varying the noise for a population with $D = 3$, $N = 10$ and $m = 10$ showed the relationship of Figure **??**. The power law relationship seems more likely than an exponential, linear, or other relationship. However, data must be obtained for larger N and m, as the effects of discretization may be obscuring the true relationship.

Varying the discretization for $N = 10$, $m = 10$ and $\epsilon = 0.2$ showed the relationship of Figure **??**. It seems roughly constant but again this needs to be checked with regard to out double exponential relationship.

4 Intergroup Dynamics

A rather different situation, that of the Nash demand game occurring between two distinguishable groups, was also considered. In these case each group consists of N agents, each with two memories of length m: one of the bids of his

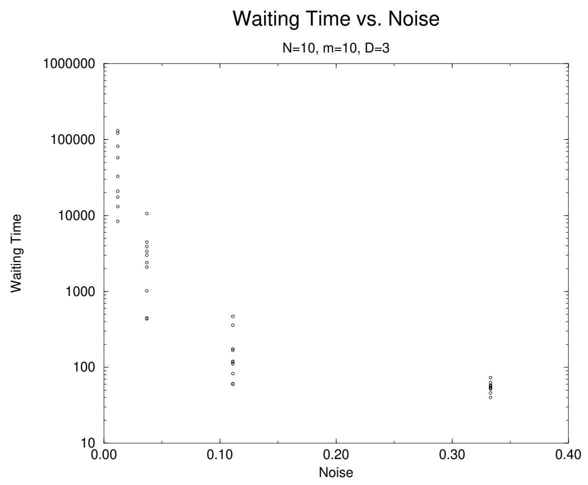

Figure 3: Waiting Time vs. Noise, $D = 3,\ N = 10,\ m = 10$.

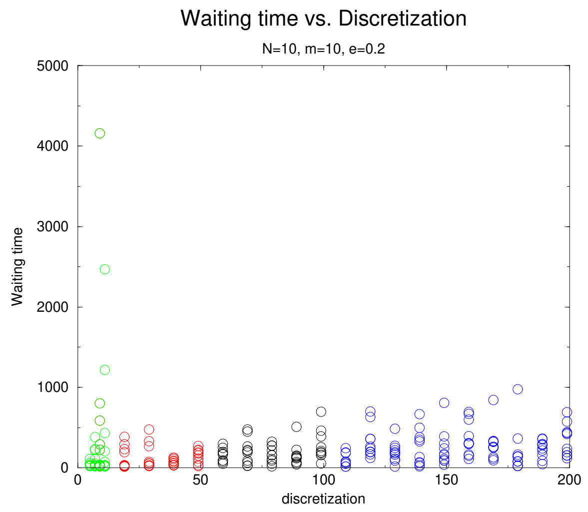

Figure 4: Waiting Time as a Function of Discretization, $N = 10,\ m = 10,\ \epsilon = 0.2$.

own group members and the other of the bids of the other group members. To isolate the intergroup dynamics, only the latter need be considered. The groups are initialized so that one group has a memory of the other group as always the highest possible bid, and the other group a memory of the first group as always the lowest possible bid. Once again, the time for the two groups to reach the 0.50-0.50 equilbrium is measured. In the absence of complete data, we can only say that the corresponding times are much longer, almost definitely by an exponential factor. The analytical difference in the dynamics is not well understood. Figure **??** shows the intergroup waiting time for low memory lengths, on a double exponential plot.

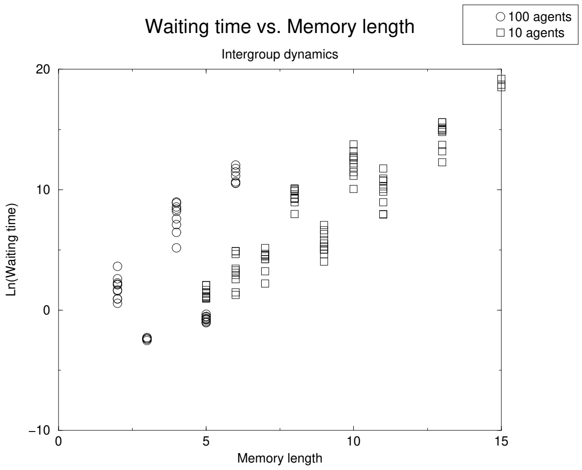

Figure 5: Intergroup Waiting Time as a Function of Memory, $D = 3,\ N = 10,\ \epsilon = 0.2$.

4.1 Noise in Agent Recognition

Another parameter was introduced to the intergroup case, allowing a different sort of experiment: noise in agent recognition. Let η be the fraction of encounters in which an agent assigns his opponent's bid to his memory of his own group or his memory of the other group with equal probability. Two situations were studied with regard to this parameter. The first was how quickly the ensemble of agents converged to the 0.50-0.50 equilibrium when $\epsilon > 0$ and η ranges from 0 to 1. When η is 0, it is the same as the intergroup waiting time. When $\eta = 1$, T is the same as that of the intragroup waiting for $2 \cdot N$ agents, each with memory length $2 \cdot m$ (but consider effect of split memory). The behavior of T as η ranges from 0 to 1 is to be determine. The second situation involved a different initialization: all agents' intragroup memories set to 0.50 and intergroup memories set to 0.75 for one group and 0.25 for the other. For $\eta > 0$ and

$\epsilon = 0$, this isolates the effect of the recognition noise. However, to truly make a comparison I should have set the intragroup memories to 0.50 in the first case as well. To be done if this section is deemed interesting enough.

5 The Continuous Case

The Continuous Case was studied to observe the limit of increasing the discretization, though it may show qualitatively different dynamics. It seems that as ϵ increases from 0 to 1 the equilibrium average memory for the single group case ranges from 0.5 to some peak and then falls back to 0.5. The location of the peak should be established, as well as the asymmetry that causes an increase in equilibrium average. Also, the relationship with respect to memory and noise simultaneously should be established, to see if the equilibrium memory actually goes to 1 as m goes to infinity, as supposed previously.

5.1 Noise as a Perturbation of the Calculated Play

A "lock-in" phenomenon was noticed in that for certain real values were played almost exclusively as time went on during certain runs. The probable reason is that each time an agent plays x, his opponent is more likely to play 1-x, and should enough agents be playing x and 1-x, these memories will be repeatedly restored in the total group memory, preventing the group from reaching 0.50. To eliminate this phenomenon, runs were performed with the noise changed to a perturbation of the calculated play: when an agent's most valuable play is x, his play is drawn from the uniform distribution on $[x - \delta,\ x + \delta]$. This indeed eliminated lock-in but still the equilibrium was slightly above 0.50, about 0.53 for $m = 60, 120, 240$, $\delta = 0.05$.