Breakdown and Recovery in Traffic Flow Models

diploma thesis of

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Abstract

In this work we compare probabilistic traffic models to a gas-liquid system. A gas-liquid system at low temperatures has two phases, the gaseous phase at low density and the liquid phase at high density. The transition from gaseous to liquid happens through phase coexistence, where both gaseous and liquid phase is found in the system. At high temperatures, these two phases become more and more similar. Above a critical temperature there is only one phase. At that temperature there is a critical point where spatial correlations go to infinity.

In terms of traffic models, the gaseous phase corresponds to laminar flow and the liquid phase corresponds to a jam. Instead of the temperature, the traffic models have a parameter that determines the amount of randomness. We analyse the Krauss-model and a slow-to-start version of the Nagel-Schreckenberg cellular automaton. We show that they have a two phase regime at low noise values and a one phase regime for high noise. We establish the coexistence curve that bounds the area of phase coexistence in the density/noise space.

Chapter 1

Introduction

Both from an operations and from a design perspective, the capacity of a road is an important quantity. Clearly, if demand exceeds capacity, queues will form, which represent a cost to the driver and thus to the economic system. In addition, such queues may impact other parts of the system, for example by spilling back into links used by drivers who are on a path that is not overloaded.

For a variety of reasons, however, capacity is not a deterministic fixed quantity. It is possible that one day a queue forms and the next day not, and this may even happen in spite of demand being larger on the second day. In consequence, any definition of capacity needs to take its stochastic nature into account.

For example, one could measure flow in 15-min intervals, say from 6am to 6:15am, from 6:15am to 6:30am, etc. One could then take the daily maximum of these values, and average the result over many typical workdays.

As an alternative, one could measure flow as a function of density irrespective of any other state variables. One could then obtain the average flow for each density interval, and the maximum of these flow-values would represent capacity.

All of these measurements have the property that they result in an expected value, i.e. in a number that, for a given day, can be exceeded or not be reached. In consquence, it is useful to develop models of traffic which reflect the stochastic nature of traffic. Clearly, the stochasticity can come at many different levels: demand can vary; road conditions can vary; driving behavior can vary; etc. These different contributions to stochasticity will have different influences, which need to be debated. In this work, we want to concentrate on road capacity. We understand that there is active research to eventually include aspects of stochastic transitions into the Highway Capacity Manual [1].

Traffic models can be distinguished on whether they show one or two phases. In a 1-phase model, the throughput of a road is uniquely determined by the traffic density. Jams cannot exist in such a model, any initial jam will "smear out" and thus eventually go away, even with unchanged traffic conditions.

In a 2-phase model, theory predicts that there can be a hysteretic transition from laminar flow to a state with jams without a change in density. This means that, at a given density, traffic can operate in the laminar flow state for long times, until it will eventually "break down" and a jam emerges. Jams are stable, they do not just go away as in a 1-phase model.

1-phase and 2-phase models are also known in physics, e.g. the gas-liquid transition. If a gas is compressed at low temperatures, it will at some point

start to condensate: small droplets of liquid form. When the gas is further compressed, these droplets grow until finally only liquid remains. If we substitute gas with laminar flow and liquid with jam, the described transition corresponds to a 2-phase traffic model. If on the other hand the gas is compressed at high temperature, no droplets form. The gas molecules just get closer and closer together. This corresponds to a 1-phase model. Thus, the temperature determines whether the gas-liquid system has one or two phases.

We investigate two probabilistic traffic models: The Krauss-model and a slow-to-start version of the Nagel-Schreckenberg cellular automaton. These models have a parameter that defines the amount of randomness or noise. We show that this parameter corresponds to the temperature in the gas-liquid system: With little noise, these models show two phases, whereas with high noise, they have only one phase.

Chapter 2

Principals of thermodynamics

Before studying the gas-liquid transition, we need some essential thermodynamic definitions and theorems. They are stated here without proof. For a comprehensive discussion see e.g. [7].

We consider an assembly of N molecules restrained in a volume V. Its energy is E and the entropy S. The volume per molecule is $v = V/N$ and the entropy per molecule is $s = S/N$. The pressure P is defined as

$$P = -\frac{\partial E}{\partial V} \tag{2.1}$$

and the temperature T

$$\frac{1}{T} = \frac{\partial S}{\partial E} \tag{2.2}$$

According to this definition, the unit of the temperature is Joule. Usually, temperature is given in Kelvin. These two measures are related through Boltzmann's constant k:

$$k T_{\text{Kelvin}} = T_{\text{Joule}}$$
$$k = 1.3806 \cdot 10^{-23} \text{J/K}$$

For easier reading of the formulas, we give temperatures in Joule.

The chemical potential μ relates the energy of the system to the number of particles:

$$\mu = \left(\frac{\partial E}{\partial N}\right)_{S,V} \tag{2.3}$$

where the subscript S, V indicates that the derivative is to be taken at constant entropy and volume. Changes in the chemical potential are related to temperature and pressure change by

$$d\mu = s dT + v dP \tag{2.4}$$

If a system is in equilibrium, the temperature is equal in every part of the system. The same holds for the pressure and the chemical potential.

The free energy F is defined as

$$F = E - TS \tag{2.5}$$

The free enthalpy Φ is

$$\Phi = E - TS + PV$$

For a system to be in equilibrium, Φ must be minimal, i.e.

$$\delta E - T\delta S + P\delta V > 0 \tag{2.6}$$

It can be shown that this implies

$$\left(\frac{\partial P}{\partial V}\right)_T \leq 0 \tag{2.7}$$

This means that increasing the volume at constant temperature will decrease the pressure.

The thermodynamic variables such as P, T, μ are actually random variables. Their values usually are so close to their average values[1] that this is often neglected. Nevertheless, there are small deviations from the average, the variables *fluctuate*. If $R_{\min}$ is the minimal work required to move the system from its average state to some state with fluctuations, then the probability to find the system in that state is

$$w \sim \exp\left(-\frac{R_{\min}}{T}\right) \tag{2.8}$$

[1] average values are denoted by a bar overhead, e.g. $\bar{T}$

Chapter 3

Phases and phase coexistence

The state of a homogeneous body in equilibrium is determined by two thermodynamic variables, e.g. P, T. However, at given P, T the body may not be homogeneous but decompose into two homogeneous parts. These parts are in different states, e.g. they have different densities. These states that can exist together in equilibrium and touch each other are called *phases*. When the system decomposes into two phases, this is called *phase coexistence*. If the entire system is in the gaseous phase, we say that the system is in the gaseous state. If there is only liquid, we call it the liquid state. A system with both gas and liquid is in the coexistence state. In this section we first describe the gaseous and the liquid phase and then consider phase transitions which will lead us to phase coexistence.

The gaseous phase is found at low density. Distances between particles vary, but the probability of having two particles close to each other is very small. If interactions between particles are completely ignored, the gas is called ideal and satisfies Claperon's equality:

$$P\,V = N\,T \tag{3.1}$$

When the gas is compressed, the particles get closer together and interactions between particles become more important. The real gas deviates more and more from the idealisation of (3.1). At some point, the gas condensates and becomes a liquid.

In the liquid phase, the molecules are close together. There is no crystalline structure as in solids, but the density is similar and in some cases (e.g. in water) even higher in the liquid than in the solid phase. Because of the fact that the particles are so close to each other, it is difficult to compress the fluid any further. A general description such as (3.1) is not possible for the liquid phase, because the interactions between the particles depend on the molecules constituting the liquid.

We now try to describe the gas-liquid transition. Such a description cannot be numerically precise, but it should describe the transition qualitatively correct. Furthermore, it should be correct for the extreme cases: For low density gases it should become Claperon's equality (3.1) and for high density it should show the

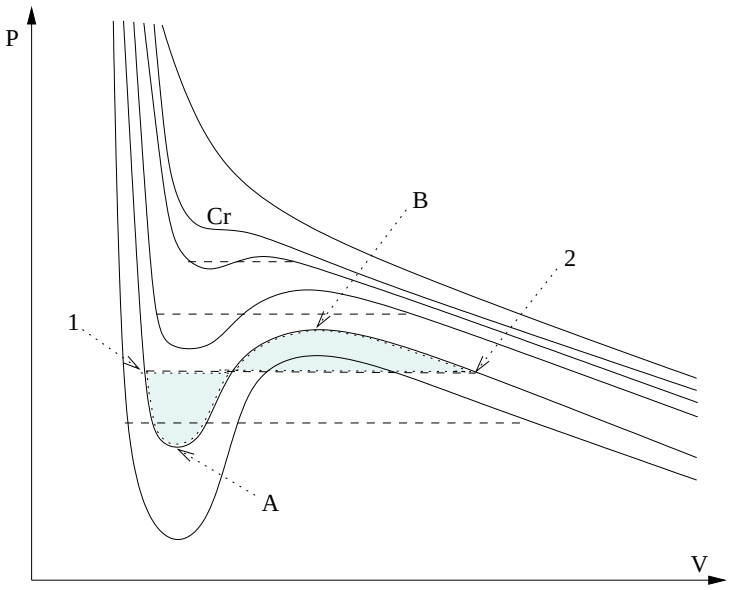


Figure 3.1: Isotherms of Van der Waal's equality. "Higher" isotherms correspond to higher temperature.

limited compressibility of liquids. Van der Waal's equality has these properties. It is

$$P = \frac{NT}{V - Nb} - \frac{N^2 a}{V^2} \tag{3.2}$$

a and b are parameters.

Fig. 3.1 shows some isotherms (curves of constant temperature). Observe that the slope can become positive (e.g. in the curve A-B), i.e. $\left(\frac{\partial P}{\partial V}\right)_T > 0$. This violates the equilibrium condition (2.7), thus that state cannot exist in nature. Instead, some part of the system is in the gaseous phase and the rest in the liquid phase (phase coexistence). Connecting the points A and B for all isotherms, we find the *spinodal curve* of Fig. 3.2. Within it, the system cannot be homogeneous.

Let the system be composed of two parts, part 1 is in the gaseous phase and part 2 in the liquid phase. For this phase coexistence to be stable, temperature, pressure and chemical potential must be equal in both parts:

$$\begin{aligned} T_1 &= T_2 \\ P_1 &= P_2 \\ \mu_1 &= \mu_2 \end{aligned}$$

With Fig. 3.1 in mind, the first equality says that 1 and 2 are on the same isotherm. The second condition implies that 1 and 2 can be connected by a horizontal line. Thus, phase coexistence must happen in a larger area than A-B. The heights of the line 1-2 is defined by the third condition, which can be

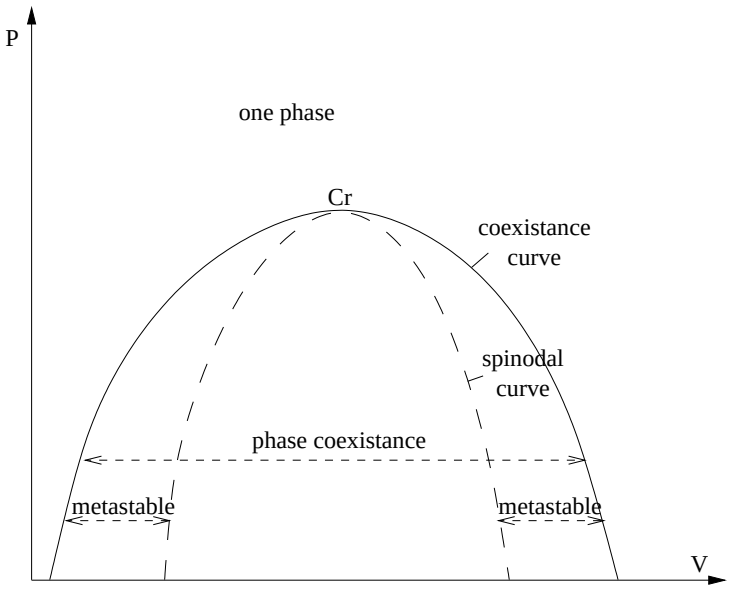


Figure 3.2: Coexistence curve and spinodal curve in the $P-V$ diagram.

written as $\int_1^2 d\mu = 0$. If we integrate along the isotherm, we get with (2.4)

$$\int_1^2 V\,dP = 0$$

This means that the two shaded areas must have the same size. Connecting the points 1 and 2 for all isotherms, we find the *coexistence curve* of Fig. 3.2. Crossing this curve corresponds to a *first order phase transition.*

For high temperatures, there is no phase transition. It is not possible to distinguish between gas and liquid. There is only one phase, which we call the *high temperature phase.*

To summarise this section, consider a gas that is slowly compressed at low temperature (see Figure 3.3). The molecules get closer together and the pressure increases. At some point, small droplets of liquid emerge. Further compressing the gas will not increase the pressure, instead the droplets grow until there is only liquid. Now the pressure increases very quickly when further decreasing the volume.

In a P-T diagram, the area of phase coexistence is only a curve given by $\mu_1(P,T) = \mu_2(P,t)$. This line may end in a *critical point.* This point is also found in the P-V diagram where the curve for the liquid⇒gas transition (containing 1) and the curve for the gas⇒liquid transition (containing 2) meet. In this point $\left(\frac{\partial P}{\partial V}\right)_T = 0$. We analyse the critical point in the next section.

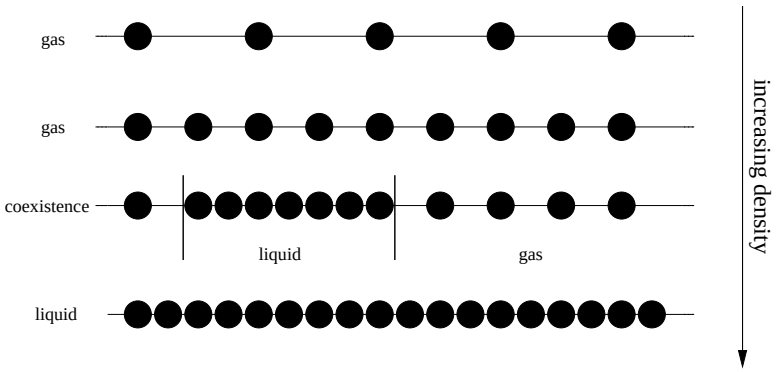


Figure 3.3: Schematic representation of the gas-liquid phase transition in one dimension

Chapter 4

The critical point

We have seen that at the critical point

$$\left(\frac{\partial P}{\partial V}\right)_T = 0 \tag{4.1}$$

we now investigate the equilibrium condition (2.6) for this case. For $T = \text{const}$ we have

$$\delta E + P\delta V > 0$$

We expand δE for small δV. The first term $\left(\frac{\partial E}{\partial V}\right)_T \delta V = -P\delta V$ (by definition (2.1)) cancels away. For the second term we have $\frac{1}{2}\left(\frac{\partial^2 E}{\partial V^2}\right)_T = -\frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_T = 0$ according to (4.1). Thus,

$$\frac{1}{3!}\left(\frac{\partial^2 P}{\partial V^2}\right)_T \delta V^3 + \frac{1}{4!}\left(\frac{\partial^3 P}{\partial V^3}\right)_T \delta V^4 + \cdots < 0$$

must be satisfied for every δV. This requires that

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0, \qquad \left(\frac{\partial^3 P}{\partial V^3}\right)_T \leq 0 \tag{4.2}$$

Let $n = N/V$ be the particle density. The critical point is at $T = T_{\text{cr}}$, $P = P_{\text{cr}}$ and $n = n_{\text{cr}}$. For the distance to the critical point let

$$t = T - T_{\text{cr}}, \qquad p = P - P_{\text{cr}}, \qquad \eta = n - n_{\text{cr}}$$

In these variables, condition (4.1) and (4.2) are

$$\left(\frac{\partial p}{\partial \eta}\right)_t = 0, \qquad \left(\frac{\partial^2 p}{\partial \eta^2}\right)_t = 0, \qquad \left(\frac{\partial^3 p}{\partial \eta^3}\right)_t \geq 0, \qquad \text{for } t = 0$$

Using these conditions, the dependence of the pressure from the temperature and the density close to the critical point is

$$p = bt + 2at\eta + 4B\eta^3 \tag{4.3}$$

with constants a, b and $B \geq 0$. For $t > 0$ all states of a homogeneous body are stable (there is no phase separation), thus according to (2.7) $(\partial p/\partial \eta)_t \geq 0$ for all η, which requires that $a \geq 0$. The isotherms of (4.3) are analogous to those from Van der Waal's equality.

Thus far, we considered the density n (and also T and P) to be constant in a homogeneous system. This of course is only an approximation, in reality there are always small fluctuations. They are measured by

$$\Delta n = \Delta n(\mathbf{r}) = n(\mathbf{r}) - \bar{n} \tag{4.4}$$

where $\bar{n}$ is the average or global density and the vector $\mathbf{r}$ the position in the system[1]. Note that $\Delta n = \Delta \eta$.

We are interested in the spatial correlation of the density, i.e. in questions like "If there is high density at $\mathbf{r_1}$, how big is the probability that the density at $\mathbf{r_2}$ is also high?". The correlation function is

$$G(\mathbf{r}) = \mathrm{E}[\Delta n(\mathbf{r_1}) \Delta n(\mathbf{r_2})], \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2} \tag{4.5}$$

We now approximate $G(\mathbf{r})$ close to the critical point for large $\mathbf{r}$. For this we need (2.8) ($w \sim \exp(-R_{\min}/T)$) where we use $R_{\min} = \Delta F$. Thus we first approximate ΔF for small Δn and then analyse w for different wavelengths of Δn. The result is equation (4.15), so if you are not interested in the derivation, you can continue reading there.

When the density fluctuates locally, the energy of the entire system fluctuates[2]:

$$\Delta F_{\mathrm{tot}} = \int (F - \bar{F}) dV \tag{4.6}$$

F here is the free energy per volume. ΔF_{tot} depends on Δn, for which we will expand it at constant temperature:

$$F - \bar{F} = \left(\frac{\partial F}{\partial n}\right)_T \Delta n + \frac{1}{2}\left(\frac{\partial^2 F}{\partial n^2}\right)_T (\Delta n)^2 \tag{4.7}$$

The first term cancels when integrating, because the total number of particles is constant. Analyse the second term: Because F refers to a volume and not the number of particles, $(\partial F/\partial n)_T = \mu$. . Then,

$$\left(\frac{\partial^2 F}{\partial n^2}\right)_T = \left(\frac{\partial \mu}{\partial n}\right)_T = \frac{1}{n}\left(\frac{\partial P}{\partial n}\right)_T$$

the second equality follows from (2.4), which becomes $d\mu = v dP$ for T=const.

For $T < T_{\mathrm{cr}}$, the system is inhomogeneous and F depends on the gradient of Δn and higher directional derivatives. Terms of the form $f(n)\partial n/\partial x_i$ become a surface effect when integrating over the volume, in which we are not interested. The same is true for the terms const $\cdot \frac{\partial^2 n}{\partial x_i \partial x_j}$. Thus, only the terms $n\frac{\partial^2 n}{\partial x_i \partial x_j}$

[1] $n(\mathbf{r})$ actually is the density of some area around $\mathbf{r}$. This area must be large enough that quantum mechanical effects can be ignored.

[2] We consider an open system, i.e. it can exchange energy with the environment. For an isolated system, the total energy is constant and cannot fluctuate

and $\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_j}$ have to be considered. Integration over the volume transforms the former into the latter. Putting all of the above together, we get for (4.6)

$$\Delta F_{\rm tot} = \int \left\{ \frac{1}{2n_{\rm cr}} \left(\frac{\partial p}{\partial n}\right)_t (\Delta n)^2 + g \left(\frac{\partial \Delta n}{\partial \mathbf{r}}\right)^2 \right\} dV \tag{4.8}$$

We now perform a Fourier expansion on Δn:

$$\Delta n = \sum_{\mathbf{k}} \Delta n_{\mathbf{k}} e^{i\mathbf{kr}}$$
$$\Delta n_{\mathbf{k}} = \frac{1}{V} \int \Delta n e^{-i\mathbf{kr}} dV$$

Since Δn is real, $\Delta n_{-\mathbf{k}}$ is the conjugate complex of $\Delta n_{\mathbf{k}}$. We get

$$\Delta F_{\rm tot} = \frac{V}{2} \sum_{\mathbf{k}} |\Delta n_{\mathbf{k}}|^2 \phi(\mathbf{k}) \tag{4.9}$$

$$\phi(\mathbf{k}) = \frac{1}{n_{\rm cr}} \left(\frac{\partial p}{\partial n}\right)_t + 2g\mathbf{k}^2 = \frac{2}{n_{\rm cr}}(at + 6B\bar{n}^2) + 2g\mathbf{k}^2 \tag{4.10}$$

To get the last equality, (4.3) was inserted.

We now insert this into (2.8) with $R_{\rm min} = \Delta F_{\rm tot}$ and get

$$w \sim \exp\left(-\frac{\Delta F_{\rm tot}}{T}\right) = \prod_{k>0} \exp\left(-\frac{V}{T}|\Delta n_{\mathbf{k}}|^2 \phi(\mathbf{k})\right) \tag{4.11}$$

Thus, the fluctuations for different wavelengths are statistically independent. Their variance is

$$\mathrm{E}[|n_{\mathbf{k}}|^2] = \frac{T}{V\phi(\mathbf{k})} \tag{4.12}$$

With this result we return to the correlation function, which can be written as

$$G(\mathbf{r}) = \sum_{\mathbf{k}} \mathrm{E}[|n_{\mathbf{k}}|^2 e^{i\mathbf{kr}}] \tag{4.13}$$

Approximating the sum by an integral and assuming three dimensional space yields

$$G(\mathbf{r}) = \int \mathrm{E}[|n_{\mathbf{k}}|^2] e^{i\mathbf{kr}} \frac{V d^3k}{(2\pi)^3} \tag{4.14}$$

The integral evaluates to[3]

$$G(r) = \frac{T_{\rm cr}}{8\pi g r} \exp\left(-\frac{r}{r_{\rm c}}\right) \tag{4.15}$$

[3] use the general formula (without proof)

$$\int \frac{e^{i\mathbf{kr}}}{\xi^2 + k^2} \frac{d^3k}{(2\pi)^3} = \frac{e^{-\xi r}}{4\pi r}$$

with the correlation radius

$$r_c = \sqrt{\frac{g n_{cr}}{at + B\bar{\eta}^2}} \tag{4.16}$$

This is the grand result of this chapter. At the critical density ($\bar{\eta} = 0$) the correlation radius grows as

$$r_c \sim t^{-1/2}$$

which diverges at the critical temperature, where

$$G(r) \sim r^{-1}$$

This result is due to L. D. Landau and the theory is called Landau-theory or mean-field theory. A more general theory is based on *critical exponents*. There

$$G(r) \sim r^{-(d-2+\zeta)} \cdot \exp\left(-\frac{r}{r_c}\right) \tag{4.17}$$

$$r_c \sim |t|^{-\nu} \tag{4.18}$$

d is the dimension of the space (usually 3). The critical exponents are ζ and ν (in the Landau theory $\zeta = 0$ and $\nu = 1/2$). If ζ is not integral, the system at $t = 0$ is a *fractal*.

Close to the critical point there are long range correlations. Long temporal correlations are associated with them. Thus, if a system close to the critical point is quenched away from equilibrium, it will take very long for the system to return to equilibrium. This is known as *critical slowing down*.

Chapter 5

Metastability

In the area between the coexistence curve and the spinodal curve, there is phase coexistence although on the Van der Waal's curve, the equilibrium condition $\left(\frac{\partial P}{\partial V}\right)_T < 0$ is not violated. In fact the homogeneous solution is also possible. It corresponds to a local minimum of the energy, the global minimum is attained for phase coexistence. The homogeneous state is *metastable*.

Consider a homogeneous system in the metastable area. The transition into phase coexistence happens when fluctuations form small areas of the new phase, so called *nuclei*. These nuclei can grow until the equilibrium-ratio between the two phases is attained. We now compute the probability that a nucleus emerges and then analyse how it grows.

According to equation (2.8) the probability w of the emergence of a nucleus is proportional to $\exp(-R_{\min}/T)$, where $R_{\min}$ is the minimal work required to form the nucleus. Since temperature and chemical potential are equal in the two phases, this work is given by the change in the thermodynamic potential $\Omega = -PV$. Let the nucleus have volume V' and pressure P'. Prior to nucleation the volume of the metastable phase is $v+V'$ and the potential $\Omega = -P(V+V')$. After nucleation the surface $\mathfrak{s}$ between the two phases has to be considered. In general, the work dR to generate an infinitesimal surface of area $d\mathfrak{s}$ is

$$dR = \alpha d\mathfrak{s}$$

where α is the *coefficient of the surface tension*. With this formula the potential is $\Omega = -PV - P'V' + \alpha\mathfrak{s}$. Without gravity, the nucleus will be a sphere. If the radius is r, we have $V' = 4/3\pi r^3$ and $\mathfrak{s} = 4\pi r^2$. For the nucleus to be in equilibrium with the metastable phase, the radius must satisfy (without proof)

$$r_c = \frac{2\alpha}{P' - P}$$

finally we get the minimal work to create a nucleus of this radius:

$$R_{\min} = \frac{16\pi\alpha^3}{3(P' - P)^2}$$

If the system is only "slightly metastable", i.e. P' close to P, $R_{\min}$ becomes large and it may take very long until a random fluctuation pushes the system into phase coexistence.

For nuclei with arbitrary radius r, $R_{\min}$ is (without proof)

$$R_{\min} = -\frac{8\pi r^3 \alpha}{3r_c} + 4\pi a^2 \alpha \tag{5.1}$$

This function has a maximum at $r = r_c$. For $r < r_c$, it is energetically better for r to decrease. The nucleus will shrink and finally disappear. For $r > r_c$ the nucleus can grow.

For small r, (5.1) can be approximated by

$$R_{\min} = \frac{4\pi}{3}\alpha r_c^2 - 4\pi\alpha(r - r_c)^2$$

and we get the following probability distribution for the emergence of nuclei:

$$f_0(r) = f_0(r_c) \exp\left(\frac{4\pi\alpha}{T}(r - r_c)^2\right) \tag{5.2}$$

$$f_0(r_c) = \text{const} \cdot \exp\left(-\frac{4\pi\alpha r_c^2}{3T}\right) \tag{5.3}$$

It can be shown that the number of "surviving" nuclei (i.e. nuclei that grow above r_c) per time-unit and volume is proportional to $f_0(r_c)$.

With the development of the new phase, the original phase becomes more stable (the state moves towards the coexistence curve) and the radius r_c increases. Eventually, hardly any new nuclei are created. Furthermore, the already existing small nuclei start to dissolve again. Only the big nuclei continue to grow until in the end there is only one nucleus left. This process is called *coagulation*. One finds that the average radius of a nucleus grows proportional to $t^{1/3}$, whereas the number of nuclei is proportional to $1/t$.

Chapter 6

Krauss-model

We now move from thermodynamics to traffic models. Assume you are driving on a long, straight road. You know how fast you are (v_f), you see how fast the car in front of you is (v_l) and how far away it is (g). The gap is $g = \Delta x - l_c$, where Δx is the front-buffer-to-front-buffer distance, and l_c is the space the car occupies in a tight jam ($\approx$ 7.5m). For simplicity, the unit of the space coordinates is set to the l_c, such that $l_c = 1$. Based on v_f, v_l and g you decide whether to accelerate or to brake. You may see more things, such as in the rear view mirror the car behind you. Usually you do not care too much, as he is responsible not to crash into you, even if you brake as hard as you can. This imposes a maximal speed (v_{safe}) for each driver, at which she can still drive safely. The other major restriction for your travelling speed – apart from the power of your engine – is the speed limit ($v_{\max}$).

A *microscopic* traffic model is an algorithm describing the driver. The goal is not to give an optimal driving strategy, because a human behind the wheel does not drive optimal. The rules of the model are very simple; the complexity comes from the interactions of the cars.

Describing a driver is much easier when the time is discretized ($t = t_0, t_1, \ldots$). At time t_i, each car computes a speed it will drive until t_{i+1}. All cars do this computation at the same time, which is known as *parallel update*. Drivers cannot react to events between t_i and t_{i+1}. This models the human reaction time, which is about one second. This time becomes the unit of time and $t_i = i$.

Let a be the acceleration rate of a car. An optimal driver would drive with $v_{\text{des}} = \min(v_{\max}, v_{\text{safe}}, v+a)$, where v is the current speed. Human imperfections can be modelled by subtracting from v_{des} a random number between zero and $a\epsilon$ (but preventing driving backwards). ϵ is a free parameter.

In the computation of v_{safe} a third parameter b is introduced. In the limit of $\epsilon = 0$, b is the braking capability. For a derivation of the formula see for example [2]. The whole algorithm is

$$v_{\text{safe}} = v_l + \frac{2b(g - v_l)}{2b + v_f + v_l} \tag{6.1}$$

$$v_{\text{des}} = \min(v_{\max}, v_{\text{safe}}, v_f + a) \tag{6.2}$$

$$r \in_R [0, 1] \quad \text{(uniformly distributed)} \tag{6.3}$$

$$v_f^+ = \max(0, v_{\text{des}} - ra\epsilon) \tag{6.4}$$

This model was developed by Stefan Krauss [6], we call it the *Krauss-model*. The deterministic limit $\epsilon = 0$ of the Krauss model has been proven to be free of crashes. Even for $\epsilon > 0$, as was used for our studies, we never observed vehicles getting closer than their minimum distance. Note that all cars have the same parameters a, b and ϵ. It is not possible to have different car types like personal cars and trucks on the same road.

When simulating a traffic model on a computer, you are restricted to roads of finite length L. The question arises what to do at the start and end of the road. The common answer is to bend the road into a closed ring, which results in *periodic boundary conditions*. Let N be the number of cars on the road. The (global) density ρ then is

$$\rho = \frac{L}{N} \tag{6.5}$$

Let us summarise the parameters: From the Krauss-model, we have a, b, ϵ and $v_{\max}$. Furthermore there is L and ρ for a total of six parameters. This is too much for us to handle, so we will fix some parameters to standard values, namely:

$$\begin{aligned} v_{\max} &= 3 \\ a &= 0.2 \\ b &= 0.6 \end{aligned}$$

L we do not fix, but we are interested in the limit $L \to \infty$. This leaves us with ϵ and ρ as free parameters.

Chapter 7

Analogy to the gas-liquid system

We claim that the Krauss-model behaves similar to the gas-liquid system considered in section 3. A jam corresponds to the liquid phase and laminar flow is analogous to the gaseous phase. The parameter ϵ plays the role of the temperature (the higher ϵ, the higher the temperature). For the density the analogy is trivial.

We show that for medium ϵ, the Krauss-model displays a two phase regime: At low density there is only laminar flow, the system is in the *laminar state*. At high density there is only one big jam (the *jammed state*). In between, there is phase coexistence, i.e. there are both parts of laminar flow and jams (the *coexistence state*). For high ϵ, there is only one phase (the high temperature phase). We will establish the coexistence curve, which separates these states. We also find a critical point, where spatial correlations become very large. The existence of a metastable area limited by the spinodal is likely, but beyond the scope of this work.

Chapter 8

Pictures

Before analysing the Krauss-model numerically, it is instructive to look at the space-time plots in Fig. 8.1. Space-time plots are pictures of the time evolution of the system. In Fig. 8.1, vehicles drive to the right and time goes down. Each row of pixels is a "snapshot" of the state of the road. In principle, one can reconstruct the trajectory of a particular car by connecting the corresponding pixels. At the displayed resolution this is however close to impossible and it is mostly the larger scale traffic jam structure that one observes. Traffic jams move against the direction of driving.

Four qualitatively different types of pictures can be identified:

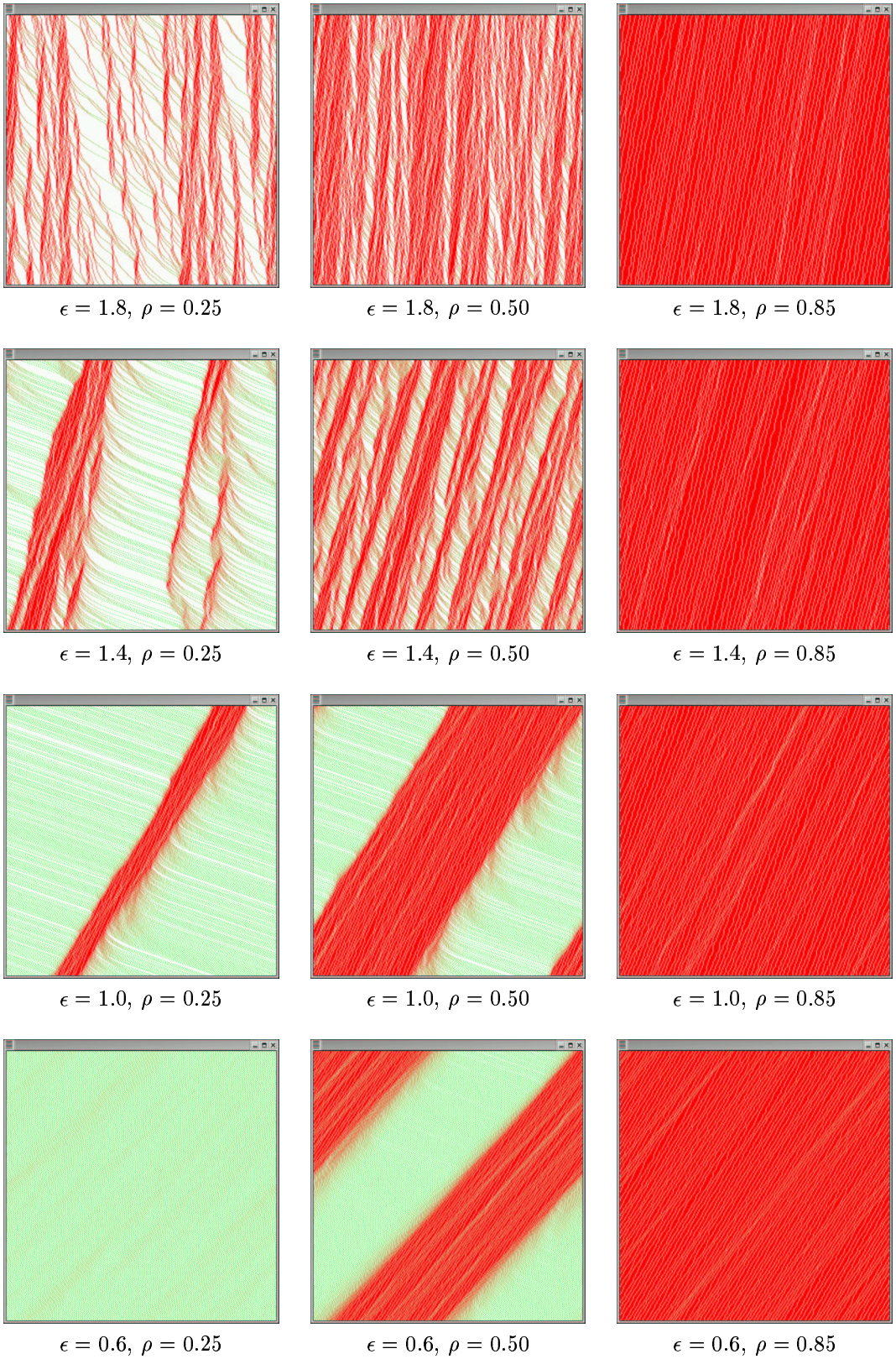
- The laminar state for ($\epsilon = 0.6, \rho = 0.25$): All cars drive at high speed. The available space is shared evenly among the cars. The traffic is very homogeneous.
- The mixed state for ($\epsilon = 0.6, \rho = 0.50$), ($\epsilon = 1.0, \rho = 0.25$) and ($\epsilon = 1.0, \rho = 0.50$): The slow cars are all together in one big jam. On the rest of the road, the cars drive at high speed. The traffic is very inhomogeneous.
- The jammed state for ($\epsilon = 0.6, \rho = 0.85$), ($\epsilon = 1.0, \rho = 0.85$) and ($\epsilon = 1.4, \rho = 0.85$): The density is so high that not a single car can drive fast. As in the laminar state, the traffic is very homogeneous.
- The single phase at high temperature for ($\epsilon = 1.4, \rho = 0.50$) and ($\epsilon = 1.8, \rho = 0.50$): Many small jams are distributed over the whole system. There is neither a larger area of free flow, nor a major jam. The traffic is homogeneous.

($\epsilon = 1.8, \rho = 0.85$) belongs to the high temperature phase, although it looks exactly like the jammed state. At very small densities, the high temperature phase looks like the laminar phase. The difference is that the transition between these two extremes is smooth. If you start from the laminar looking state and increase density, nothing "special" will happen. When the density approaches one, you are surprised to be watching a jammed looking state.

($\epsilon = 1.4, \rho = 0.25$) is harder to classify. It probably belongs to the mixed state.

Note that "homogeneous" here means "homogeneous on large scales". What this means is that there is a spatial measurement length ℓ above which all density measurements return the same value.[1]

[1] More precisely, the fluctuations from one measurement to the next are the same as in the laminar or the jammed state.



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$\epsilon = 0.6,\ \rho = 0.25$ $\epsilon = 0.6,\ \rho = 0.50$ $\epsilon = 0.6,\ \rho = 0.85$

Figure 8.1: Space-time plots for different parameters. Space is horizontal; time increases downward; each line is a snapshot; vehicles move from left to right; fast cars are green, slow cars red. Displayed are 500 time-steps for a road of length $L = 500$. The systems have run for some time before the pictures were taken, they are in equilibrium.

Chapter 9

Defining a jam

In the discussion of Fig. 8.1 we used the word jam, which has a clear intuitive meaning. Giving a precise definition however is all but easy. In the literature, there are multiple definitions:

- "A connected structure of vehicles, travelling at a velocity below a given threshold v_{thres} will be called a jam, if this structure contains at least one stopped vehicle." ([6], p.38)
- "... a car is considered to be jammed, if more than half of the n nearest neighbours (including the car itself) have got a velocity below v_{thres}." ([6], p.60)

We use the following, very simple definition: A *jam* is a sequence of maximal length of cars driving with speed less or equal v_{thres}. The cars between two neighbouring jams are in *laminar* flow.

This definition will not always correspond to our natural understanding of the word jam. Thus, whether a car is jammed or not according to this definition is just a starting point and not the final answer.

We still need a value for v_{thres}. Ideally, the distribution of the car speeds is bimodal with one peak close to zero corresponding to the jammed cars, and a second peak close to $v_{\max}$, representing the cars in the laminar phase. Between these peaks should be a region of little probability, where v_{thres} can be placed. This is indeed true for the parameters used in Fig. 9.1. We set $v_{\text{thres}} = \frac{v_{\max}}{2} = 1.5$.

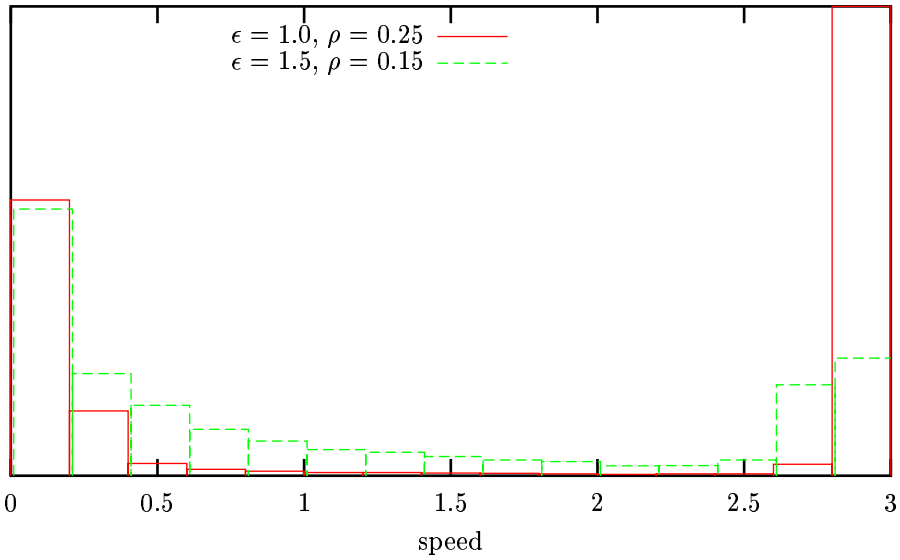

Figure 9.1: Histogram of car speeds. Measurements are made on a road with 1000 cars.

Chapter 10

Initial condition and relaxation

For most parameters of the Krauss model, there is a unique equilibrium state, which the system will attain after a finite time t_{relax}, no matter how it was started. However, deciding when the equilibrium is reached is not trivial (running the simulation for $t \to \infty$ clearly is not an option).

Let r_t be the state of the road at time t and $f(r_t)$ some property of the road(e.g. the number of jams). To find t_{relax}, we use the following idea: For small t, $E[f(r_t)]$ will depend on the initial condition. With increasing time, $E[f(r_t)]$ converges towards the equilibrium value. Assume the convergence is from above. Now we need another initial condition that approaches the equilibrium value from below. Once these two sequences are close enough together, the equilibrium value is found. Unfortunately, it cannot be guaranteed that the value thus obtained really is the equilibrium value.

We use the following two initial conditions:

- laminar: The cars are positioned equidistant over the road with speed zero.
- jammed: all cars are cramped together in a big jam without any gap. Their speed is zero.

Figure 10.1 shows space-time plots of systems started with the laminar initial condition. The jammed ($\rho = 0.85$) and the laminar ($\epsilon = 0.6, \rho = 0.25$) systems are almost immeadeately relaxed, at least the pictures look identical to those of the relaxed systems in 8.1. In the pictures for ($\epsilon = 1.0, \rho = 0.5$), ($\epsilon = 1.4, \rho = 0.5$), ($\epsilon = 1.4, \rho = 0.25$), ($\epsilon = 1.8, \rho = 0.25$) one observes coagulation: the system evolves from many small jams to few large jams. ($\epsilon = 1.6, \rho = 0.25$) shows a nasty specialty of the Krauss model . The system builds structures that resembel jams but have much lower density. These structures are highly unstable, they can either develop into real jams or disappear again. ($\epsilon = 0.6, \rho = 0.50$) is still far away from its equilibrium.

Figure 10.2 shows systems started with the jammed initial condition. For $\epsilon \leq 1$, the jam is stable. From the upstream end small holes enter the jam. They reduce the density of the jam but do not break it into two. If $\epsilon > 1$, the jam is not stable. It frays on the downstream end.

For a concrete implementation, a function f must be specified, furthermore the expected value must somehow be approximated. The latter can be done by running a number of simulations with different random seed in parallel. All simulations run for say $100'000$ time-stepsand then evaluate f. The expected value is approximated by the average of these values. To save time, the simulations can be distributed over multiple machines.

The authors did not pursue this approach, in part to avoid interprocess communication. Instead we averaged f over some time interval (but with only one simulation):

$$\mathrm{E}[f(r_t)] \approx \frac{1}{100} \sum_{i=0}^{99} f(t - 5000i)$$

This is less accurate, because the configuration of the road at time $5000i$ is not independent of the configuration at time $5000(i-1)$.

As function f we first used the number of jams on the road (see Fig. 10). Since both initial conditions start with $v = 0$, the criterion finds one large jam. Vehicles then accelerate, but because of interaction will form small jams. For that reason, the laminar start leads to many jams very quickly. From then on, the number of jams goes down, because jams coagulate. In contrast, when starting with a large single jam, than that jam remains the only one in the system for large times. In Fig. 10, we see that for $\epsilon = 1.0$, the system eventually goes to a state where it has, in the average, about 1.8 jams. In contrast, with $\epsilon = 1.5$, the system converges to an average of more than 20 jams. Also, the figure shows that the system goes to those long-run states no matter how it starts.

Once t_{relax} is determined, one is interested in the equilibrium value of some property g: $\mathrm{E}[g(r_{t_{\mathrm{relax}}})]$. To check that t_{relax} (which was determined using f) is large enough, one computes $\mathrm{E}[g(r_{t_{\mathrm{relax}}})]$ once using only systems with laminar initial condition and once using only systems with jammed initial condition. If the results are substantially different (which has to be decided by inspection) t_{relax} was too small. The success of this test gives confidence into the computed value; the way t_{relax} was attained or its value are irrelevant for the correctness.

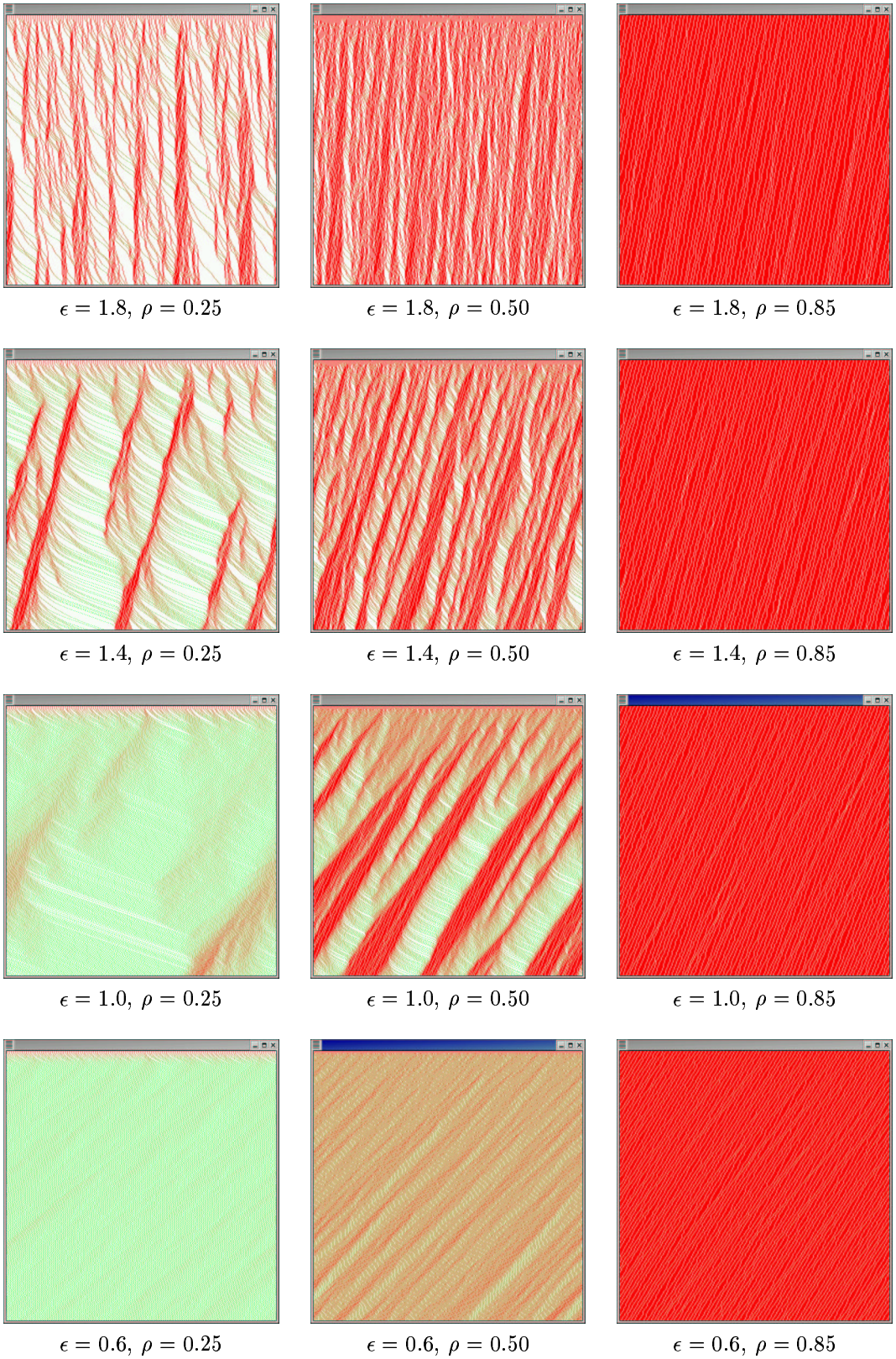

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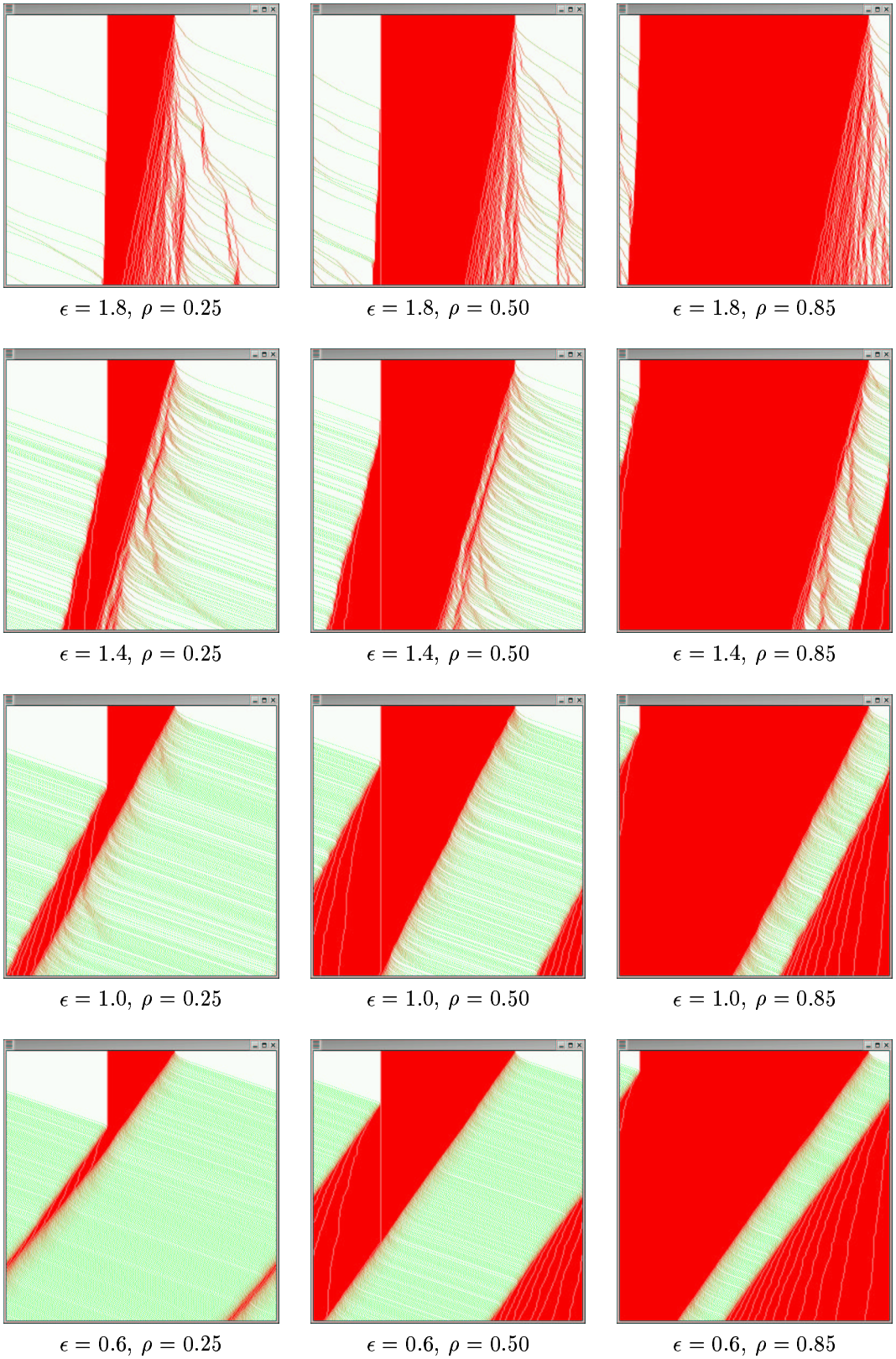
Figure 10.1: Space-time plots for systems started with the laminar initial condition. Otherwise the systems are equal to those of Fig. 8.1.



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$\epsilon = 0.6,\ \rho = 0.25$ $\epsilon = 0.6,\ \rho = 0.50$ $\epsilon = 0.6,\ \rho = 0.85$

Figure 10.2: Space-time plots for systems started with the jammed initial condition. Otherwise the systems are equal to those of Fig. 8.1.

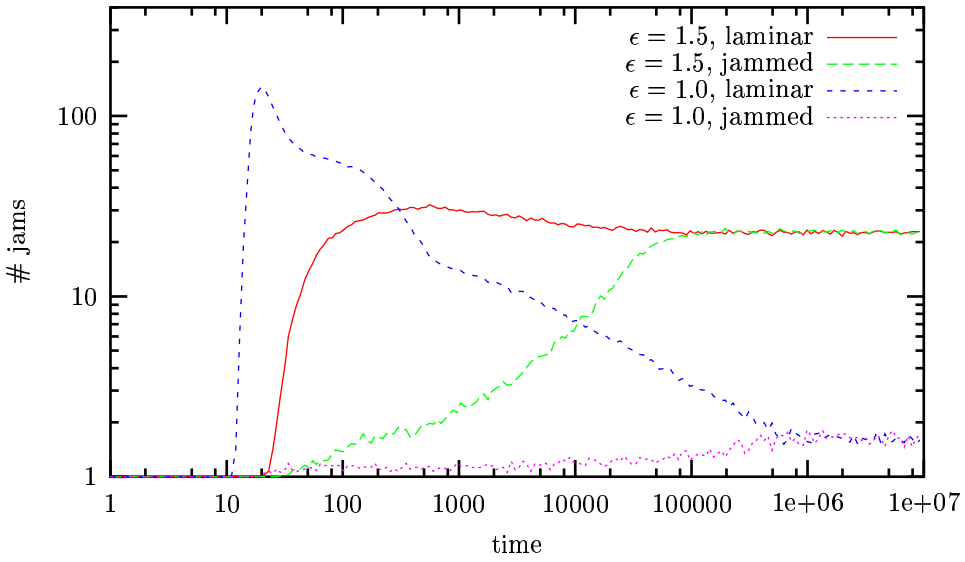


Figure 10.3: Time evolution of the number of jams. All four curves are for 1000 cars.

Chapter 11

Number of jams

In this section we investigate the number of jams in a system in equilibrium. Figure 11.1 shows this number as a function of the density ρ and the noise ϵ. The length of the road L is held constant at 4'000.

For $\epsilon \lesssim 1$ there no jam for small density, corresponding to the laminar phase. At high density there is exactly one jam spanning the entire system. This is the jammed state. In between, there are very few jams, but more than one. This might be the coexistence state, but the gas-liquid model says that all nuclei of the new phase coagulate and in the there is one compact liquid phase and one gaseous phase. The Krauss-model has multiple jams, this is not due to relaxation problems. One explanation has to do with the interface between the downstream end of a jam and the following laminar flow: In that area, the cars are the faster the farther away from the jam they are (the longer they are already accelerating). But the speed increases not monotonously, so the this area can become a sequence of small jams and laminar flows.

If ϵ is slightly larger than one, the big jam is still quite compact but there are also some small jams on the road. The latter are not stable; small jams constantly emerge, exist for some time and then dissolve again. If ϵ is substantially larger than one, the big jam disappears and there is only a number of very jams.

For ϵ close to two, the number of jams decreases again. With so much noise, cars only seldom drive faster than v_{thres}. Anyway, with Figure 8.1 in mind, a partitioning of the road in jammed and laminar parts is artificial. High ϵ corresponds to the high temperature phase of the gas-liquid model, where there is neither gas nor liquid.

At high densities, there are also less jams. Of course these jams are larger, the total number of jammed cars increases monotonously with ρ. Only a small part of the road is in laminar flow and laminar flow is needed to split a jam in two. Note the asymmetrie at $\epsilon \approx 1.6$: If there is only a small number of jammed cars, these disperse over the entire road forming many jams. If there is only a small number of fast moving cars, these tend to stay together resulting in few jams.

The number of jams depends on the system size. However, if the system has only one jam at $L = 4000$, one expects to find only one (though bigger) jam also as L goes to infinity. Those systems that have multiple jams at $L = 4000$ should have even more jams when L increases. We conjecture that the average size of a jam is independent of the system size. Since the number of jammed

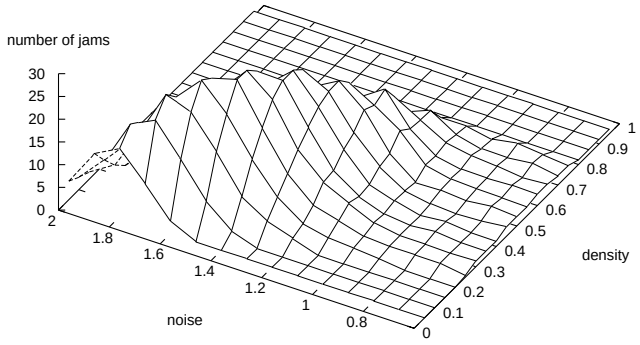


Figure 11.1: Number of jams as a function of the density ρ and the noise ϵ. $L = 4000$

cars grows linearly with L, the number of jams also has to be proportional to L.

The computer simulation results are shown in Figure 11.2. At $\epsilon = 1.5, \rho = 0.3$, our conjecture seems to hold, the measured points are very close to the straight line

$$\text{number of jams} = 0.021 \cdot N + 1$$

Keep in mind that Figure 11.2 has logarithmic axis, so a straight line does not appear straight. For $\epsilon = 1.0, \rho = 0.3$ we have already discussed one reason why there can be more than one jam and thus the average is bigger than one. Our explanation is an interface effect, and we would expect the interface (and therefore also the number of jams) to be independent of the system size. The plot however shows that the number of jams increases with L for $L > 1000$. Thus, apart from the interface effect, the noise can collaborate and form *mini-jams* within the laminar flow. A human observer probably would not consider these as real jams, but our simple definition does.

The number of jams gives some good indication to the location of the coexistence curve. But for high ϵ it is not applicable. A new measure is needed.

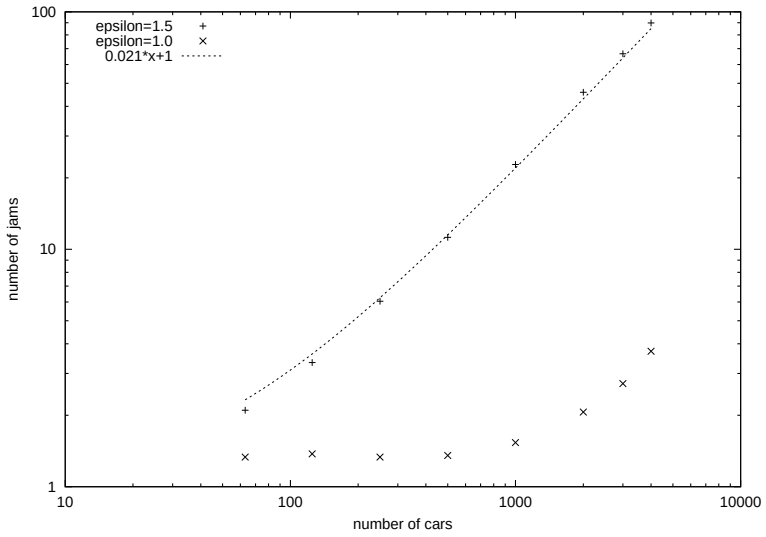


Figure 11.2: Number of jams as a function of the number $N$of cars. $\rho = 0.3$ for both roads. The line is an affine approximation of the data.

Chapter 12

Distribution of the lengths of the laminar structures

In this section we look at the lengths of the laminar structures, i.e. the distances between the jams. Figure 12.1 shows the histograms for the same parameters as the pictures of Figure 8.1.

A system in the laminar state gives a histogram one bar at length L and heights one. This means that such a road almost always has one laminar structure covering the entire road. There is also a low peak at small sizes, it is caused by the mini-jams mentioned earlier. The other extreme is the jammed state. There the histogram is empty. The road contains only one jam and no laminar flow.

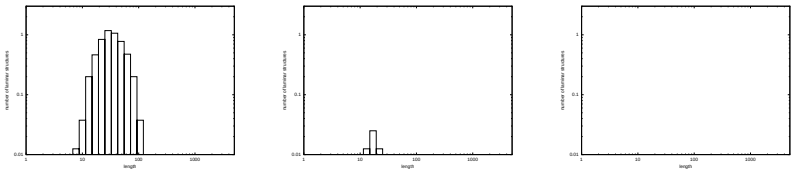
In the mixed state at moderate noise ($\epsilon \lesssim 1$) there is one bar of height one at about $L - N$. The peak at small length is caused by the interface effect and the mini-jams discussed in section 11. It is a measurement problem and not a special feature of the model. At higher noise values, the peak at $L - N$ disappears. There is no big jam here, it is broken up into many small jams.

For high ϵ (close to two) and low density, we find many small laminar structures. At higher densities, the cars can no longer accelerate above v_{thres} and there is no laminar structure. We mention again that a distinction between laminar and jammed in the high temperature phase is artificial and the application of our definition is misleading.

The histogram for $\epsilon - 1.4, \rho = 0.25$ has a special form: Laminar structures of any length are possible. Small one are more likely than large one and reduction of the expected number of structures resembles a power law:

$$\text{number of laminar structures of length } l \sim l^{\alpha}$$

This gives a straight line with slope α on a log-log plot. The parameters $\epsilon - 1.4, \rho = 0.25$ are close to the critical point, which we will investigate in section 15.



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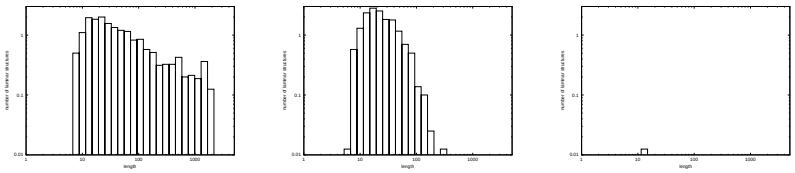

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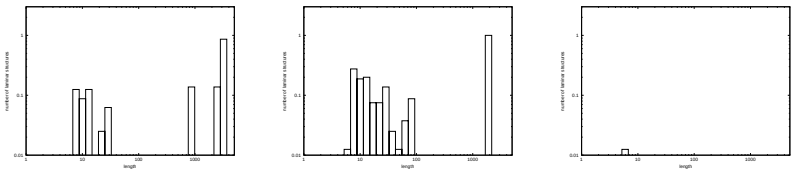

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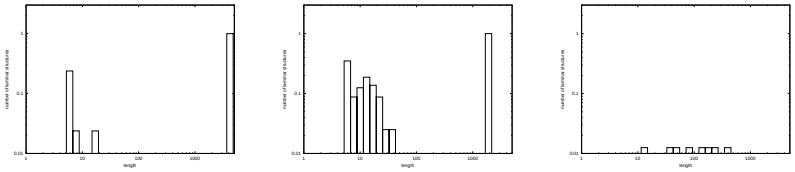


$\epsilon = 0.600,\ \rho = 0.25$ $\epsilon = 0.600,\ \rho = 0.50$ $\epsilon = 0.600,\ \rho = 0.85$

Figure 12.1: Histograms of the lengths of the laminar structures. $L = 4000$ for all plots. The heights of a bar is the expected number of laminar structures with length in the interval corresponding to the width of the bar. Plots are in log-log-scale.

Chapter 13

Density variance

The distribution of the lengths of the laminar structure investigated in the previous section contains a lot of information. But since it is a distribution and not a scalar value, it cannot be displayed in a 3d plot as a function of *rho* and ϵ. To find the coexistence curve, such a plot would help. Furthermore, all measures considered so far depend on the definition of a jam. Such a definition and specially the choice of v_{thres} is very arbitrary. It is preferable to have results that are independent of how a jam is defined. The *density variance* introduced in this section is such a measure. It should distinguish homogeneous from inhomogeneous traffic.

As already discussed earlier, it should be noted that some states that are called "homogeneous" in this paper may appear inhomogeneous to an observer. An example for this is Fig. 8.1 bottom right. As said before, these states are "homogeneous on large scales", which is the important criterion here. Essentially, this means that for system size $L \to \infty$ and measurement interval $\ell \to \infty$ (but $\ell \ll L$), all density measurements will eventually return the same value. This will not be the case for "mixed" states.

The density variance is defined as follows: Partition the road into segments of length l (for simplicity let l divide L without remainder). For each segment the local density ρ_l can be computed as the number of cars in that segment divided by l. The density variance is the variance of the local density:

$$\text{Var}[\rho_l] = \frac{1}{L/l} \sum_{i=1}^{L/l} (\rho_l(i) - \text{E}[\rho_l])^2 \ , \tag{13.1}$$

where $E[.]$ is the expected value, which in our case is the same as the systemwide density. Note that since the density lies within $[0, 1]$, the variance cannot exceed $1/4$.

What this value picks up is how much each individual measurement segment of length ℓ deviates, in terms of its density, from the average density. Assume a system consisting of jammed and laminar traffic. If there is a jam in one segment, then the segment's density will be much higher than the average density. Conversely, if there is only laminar traffic in a segment, then the segment's density will be much lower than the average density. $\text{Var}[\rho_l]$ takes the average over the square of these deviations.

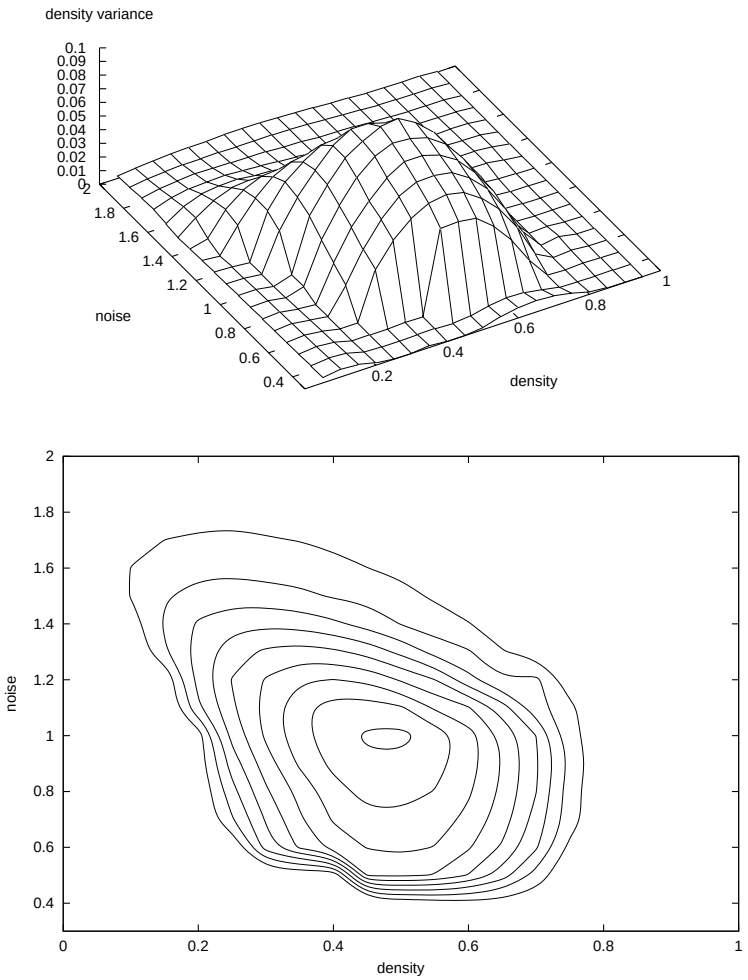


Figure 13.1: 3d-plot and isolines of the density variance. The outermost isoline is $\mathrm{Var}[\rho_l] = 0.01$, the innermost $\mathrm{Var}[\rho_l] = 0.09$. $L = 4000$ and $l = 62.5$

Fig. 13.1 shows this value as a function of the global density ρ and the noise parameter ϵ. Look at it for fixed ϵ, say $\epsilon = 1$. One sees that at densities up to $\rho \approx 0.2$, the value of $\mathrm{Var}[\rho_l]$ is close to zero, indicating a homogeneous state, which is in this case the laminar state. Similarly, for densities above $\rho \approx 0.8$, $\mathrm{Var}[\rho_l]$ is again close to zero, indicating another homogeneous state, which is in this case the jammed state. In between, for $0.2 \lesssim \rho \lesssim 0.8$, the value of $\mathrm{Var}[\rho_l]$ is significantly larger than zero, indicating a mixed state.

Now slowly increase ϵ. We see that the laminar regime ends at smaller and smaller densities, while the jammed regime starts at smaller and smaller densities. The former says that the higher the noise, the less stable the laminar flow and traffic breaks down at lower densities. The latter means that for large ϵ, the jammed phase has many relatively small holes, which reduce the density, but do not break the jam. At $\epsilon \approx 1.7$, the mixed phase completely goes away; for larger ϵ, we do not pick up any inhomogeneity at any density, which is also true for the high temperature phase in the gas-liquid model. Note that close to the transition the system still looks like it possesses different phases These structures do however exist *on small scales only*; when averaging over larger segments, then all segments contain exactly the same density. A segment length of $\ell = 62.5$, as used in the figure, is already sufficient.

The maximum of the density variance is at $\rho \approx 0.5, \epsilon \approx 1$. This can be explained as follows: $\epsilon = 1$ produces a sharp separation of the jam (high density) and the laminar structure (low density). With $\rho = 0.5$ it turns out that these two phases have the same length, thus $\mathrm{Var}[\rho_l]$ is maximal. Increasing (decreasing) ρ will increase (decrease) the length of the jam and therefore in both cases decreases $\mathrm{Var}[\rho_l]$.

The elliptical shape with its diagonal axis for $\epsilon > 1$ can be explained by the dependence of the outflow of a jam on ϵ: Consider the last car in a jam (on the downstream side). It has speed zero and enough gap to accelerate, so v_{des} in (6.3) is a. The probability that the car starts moving in the next time-step is (see (6.4))

$$P(v_f^+ > 0) = P(1 > r\epsilon) = \begin{cases} 1 & \text{for } \epsilon \leq 1 \\ 1/\epsilon & \text{for } \epsilon > 1 \end{cases} \tag{13.2}$$

The higher ϵ (for $\epsilon > 1$), the lower is this probability and the lower is the density of the laminar flow after a jam. If the laminar structures have lower density, the jams occupy more space. One needs to reduce the density in order to go back to the state where they occupy equal space.

For $\epsilon < 0.5$ traffic becomes homogeneous at any density. With so little noise, the density of the outflow of a jam is equal to the maximal density of free flowing traffic. Under these conditions, no stable jams can exist (see [6], section 5.6.3). The gas-liquid model shows no homogeneous state at low temperature, this is a special feature of the Krauss model.

If we cut Figure 13.1 at $\epsilon = 0.5$, the isothermes look very similar to the coexistence curve of Figure 3.2. They are also in accordance with the coexistence curve one might deduce from Figure 11.1. In the next section we consider a last measure from which the coexistence curve can be derived.

Chapter 14

Density of jams and laminar structures

The density of a jam is the number of cars in it divided by the length of the jam. If there is no jam, the density is set to zero. The density of a laminar structure is analogous. The measured densities are plotted in Figure 14.1. Note that these measurements depend on the definition of a jam.

The plot for the jams shows density zero at low ρ and ϵ. This is the laminar state without jams. When jams first appear, the density has a very steep slope. In the area of phase coexistance, the density of a jam is independent of the global density ρ, exactly as it is for the density of the liquid phase in phase coexistance. The increase of ρ produces larger jams, but not denser ones. Only at high ρ (in the jammed state) does the density increase. For $\epsilon > 1$ the area of constant (with regard to ρ) jam density ends for smaller and smaller ρ, when ϵ increases. These ends lie on the coexistence curve. For very high ϵ the area of constant jam density disappears, as it should in the high temperature phase.

The plot for the density of the laminar structure has the same properties: Consider a cut for constant, but not too high ϵ (e.g. $\epsilon = 1$) and sweep ρ from zero to one. First the density of the laminar structure increases (the laminar state), then remains constant (phase coexistence) and finally drops very quickly and remains zero (in the jammed state without laminar structures). The start of the constant laminar structure density moves to the left for increasing noise. The quality of the data does not allow to say much for high ϵ. As already mentioned, this is because in the high temperature phase a separation of jam and laminar flow is problematic.

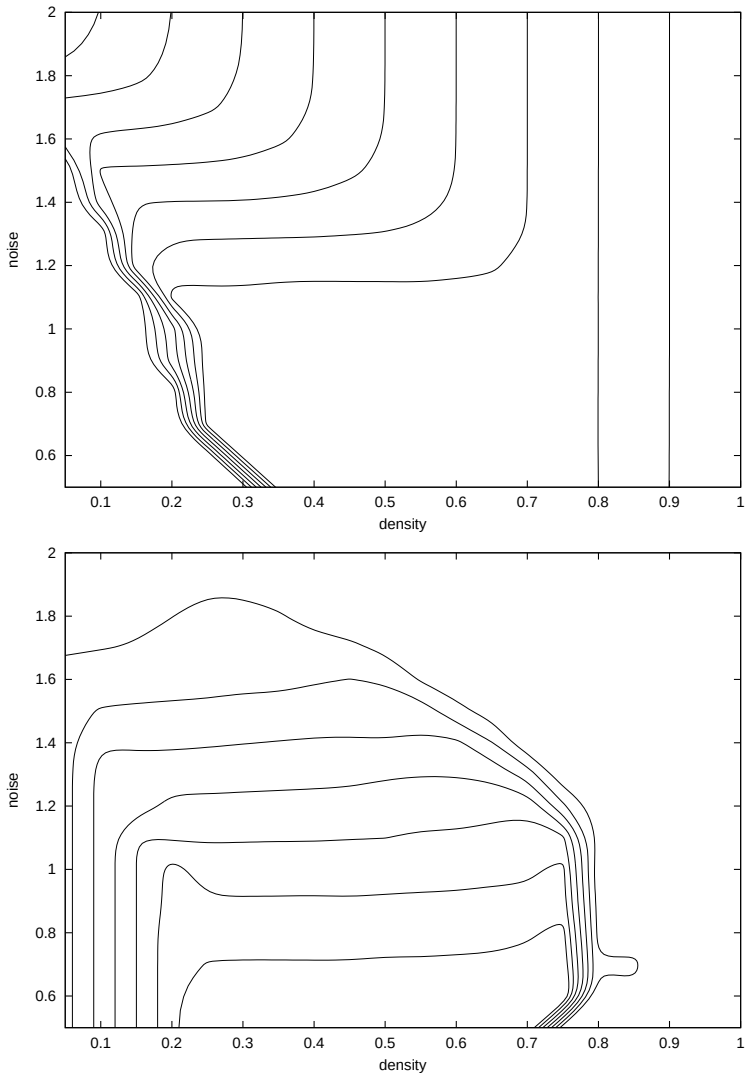


Figure 14.1: TOP: Isolines for the average density of a jam. Leftmost isoline is 0.1, rightmost 0.9. $L = 4000$. BOTTOM: Isolines for the average density of a laminar structure. Topmost isoline is 0.03, bottommost 0.21 (in steps of 0.03). $L = 4000$

Chapter 15

The critical point

In the previous chapters we showed that the Krauss-model has a two phase regime for medium noise values and a one phase regime at high noise values. The approximate location of the phase coexistence curve was also derived. From section 4 we know that at the top of the phase coexistence curve, where $\left(\frac{\partial P}{\partial V}\right)_T = 0$, there is a critical point. In this chapter, we investigate whether the Krauss model has a critical point and where it would be.

First, a criterion is needed which tells us whether a given point (in the ρ, ϵ-plane) is close to the critical point or not[1]. A central property of the critical point are the long range correlations. Recall equation (4.18):

$$G(r) \sim r^{-(d-2+\zeta)} \cdot \exp\left(-\frac{r}{r_c}\right)$$
$$r_c \sim |t|^{-\nu}$$

where $G(r)$ is the spatial density correlation defined in (4.5):

$$G(\mathbf{r}) = \mathrm{E}[\Delta n(\mathbf{r_1}) \Delta n(\mathbf{r_2})], \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$$

We do not measure $G(r)$, but a property that behaves similar, namely the distribution of the lengths of the laminar structures, which was already considered in section 12.

To see this, observe that if $n(r_1)$ is high, there is probably a jam at r_1. On the other hand, if $n(r_1)$ is small, r_1 is part of a laminar structure. Thus, if $G(r)$ is high, then the probability is high that there is laminar flow at r_2 given that at r_1 is laminar flow. It is also likely that a single laminar structure covers r_1 and r_2. So we find that

$$G(r) \sim \text{expected number of lam. str. with length greater or equal } r \quad (15.1)$$

At the critical point and for $r \gg 1$, this function should be a straight line in the log-log plot. Slightly away from the critical point, it is a straight line up to some r (where $\exp(-r/r_c)$ gets substantially smaller than one) and then the tail quickly falls to zero. The later this happens, the closer the point is to the critical point. Do not confuse the straight line for small r in all plots with a

[1] Computer simulations cannot determine the location of the critical piont exactly, which would mean with infinite precision.

power law. The function is constant there because there is no laminar structure with lenght smaller than about ten.

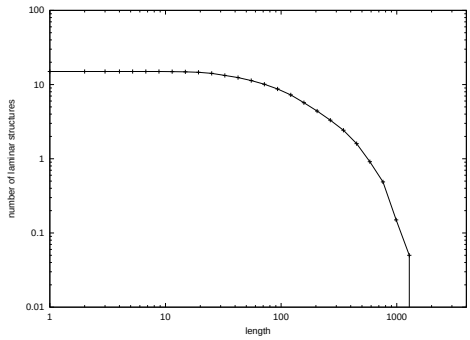
Even at the critical point, the number of laminar structures with length greater er equal r will become zero for some r because the measured system has finite length. To be precise, the critical point only exists for systems of infinite size. The larger the simulated system, the better can the critical point of the infinite system be approximated.

From Figure 13.1, we expect the critical close to $\epsilon = 1.6, \rho = 0.2$. We simulate a number of points close to that guess and apply the above criterion. Since the coexistence curve is asymmetric (it "leans to the left") the tested points with higher ϵ have smaller ρ. Figure 15.1 and 15.2 show the number of laminar structures with greater or equal r as a funtion of r. A data points at r can be derived from a histogram of Figure 12.1 by adding the heights of all bars to the right of r.

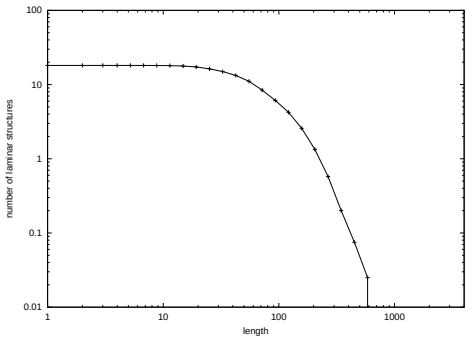
For $\epsilon = 1.8$, no power-law is visible, at least no over a reasonabley big interval. The critical point must be at lower noise. For $\epsilon = 1.7$ only $\rho = 0.05$ looks promising. The power-law holds with some accuracy for $r \in [100, 1000]$. $\epsilon = 1.6, \rho = 10$ looks even better. Also $\epsilon = 1.5, \rho = 15$ cannot be ignored. Observe how the density of these guesses decreases with increasing ϵ. This was expected, as also the coexistence cureve "leans to the left".

To get a better extimate of the critical point, the system size has to be increased. This is done for the plots in Figure 15.3 and 15.3, where $L = 32'000$. The displayed function is the same. The best looking are $(\epsilon = 1.6, \rho = 0.10)$ and $(\epsilon = 1.5, \rho = 0.20)$. In Figure 15.5 we extimate the slope of the straight line for these paramters, which is the value $-(d - 2 + \zeta)$. $d = 1$ as the road is one-dimensional. For $(\epsilon = 1.6, \rho = 0.10)$, $\zeta \approx 1.95$ approximates the data points in the interval $[200, 3000]$. For $(\epsilon = 1.5, \rho = 0.16)$ one gets with $\zeta \approx 1.9$ an approximation in $[80, 3000]$.

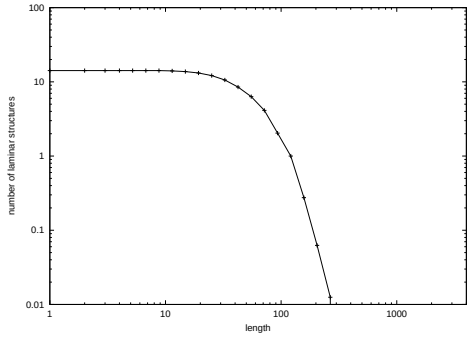
The data presented in this chapter shows that the Krauss model has an area where spatial correlations become large. It is not clear whether there is a critical point or not. Even if a critical point is present, it could be that $\zeta = 0$, so the Krauss-model would not be fractal. One way to attack this problem is to simulate more systems with different road lengths, some considerably larger than $L = 32'000$. With this data one can try to extrapolate the behaviour for $L \to \infty$. This is beyond the scope of this work.



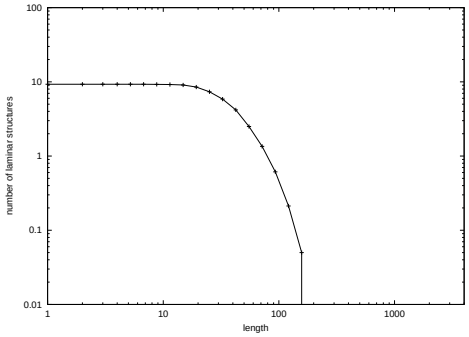
$\epsilon = 1.800,\ \rho = 0.05$



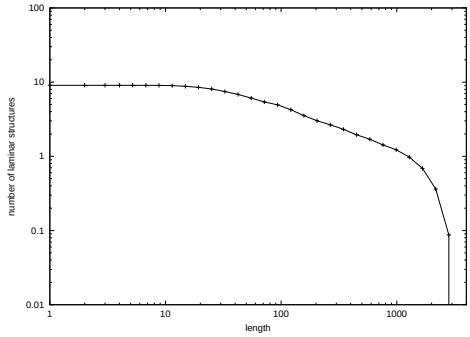
$\epsilon = 1.800,\ \rho = 0.10$



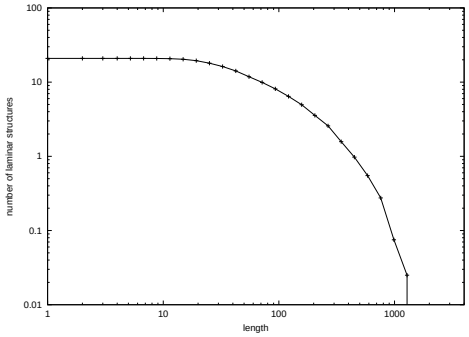
$\epsilon = 1.800,\ \rho = 0.15$



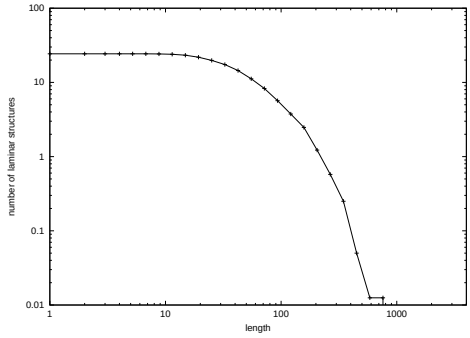
$\epsilon = 1.800,\ \rho = 0.20$



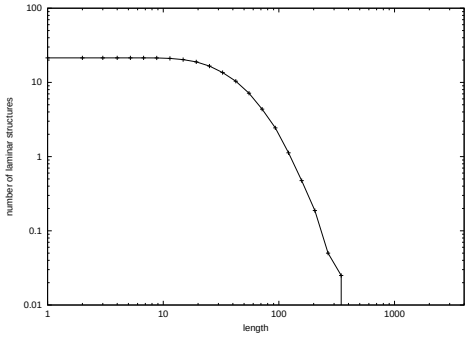
$\epsilon = 1.700,\ \rho = 0.05$



$\epsilon = 1.700,\ \rho = 0.10$

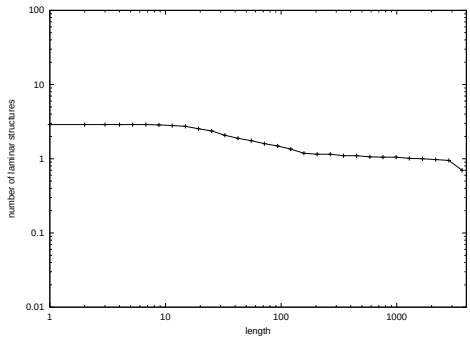


$\epsilon = 1.700,\ \rho = 0.15$

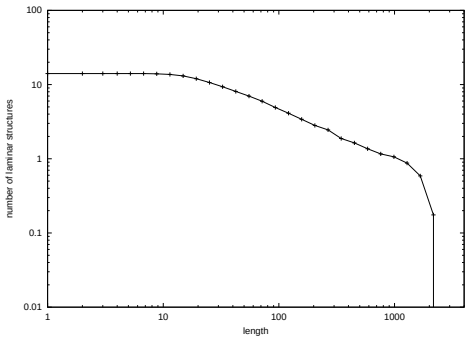


$\epsilon = 1.700,\ \rho = 0.20$

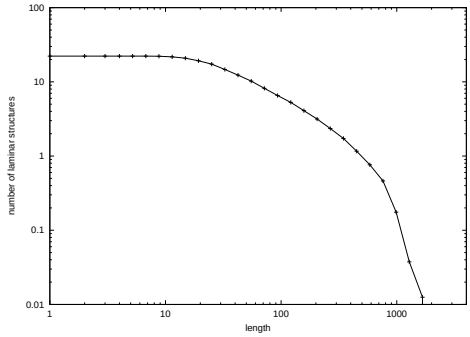
Figure 15.1: number of laminar structure with size greater or equal l



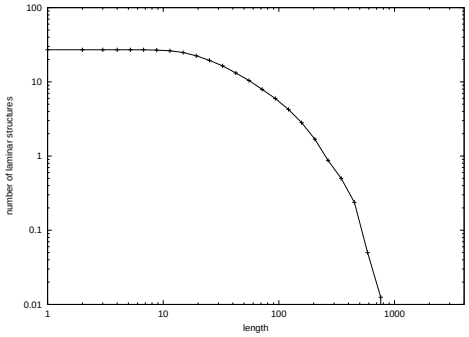
$\epsilon = 1.600,\ \rho = 0.05$



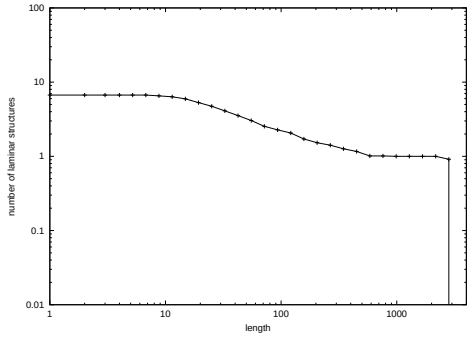
$\epsilon = 1.600,\ \rho = 0.10$



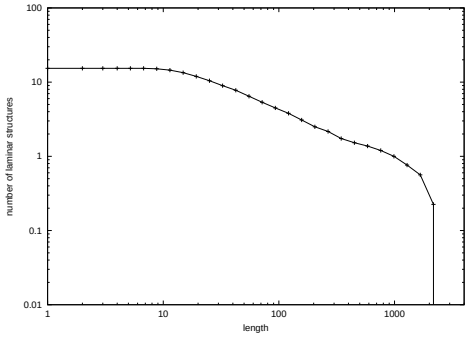
$\epsilon = 1.600,\ \rho = 0.15$



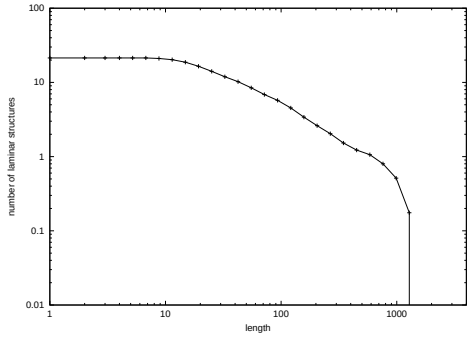
$\epsilon = 1.600,\ \rho = 0.20$



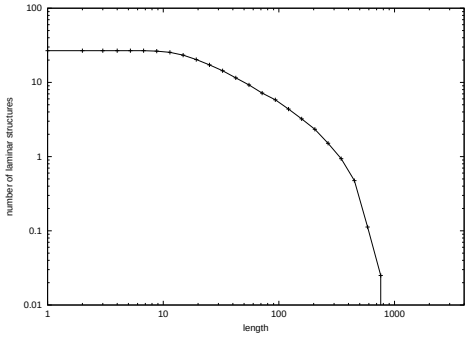
$\epsilon = 1.500,\ \rho = 0.10$



$\epsilon = 1.500,\ \rho = 0.15$

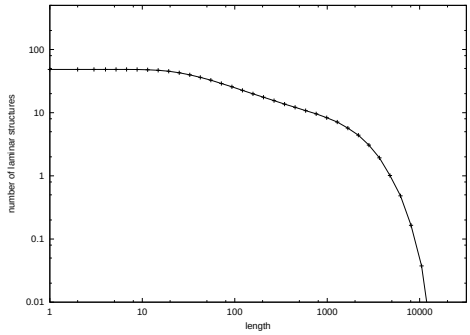


$\epsilon = 1.500,\ \rho = 0.20$

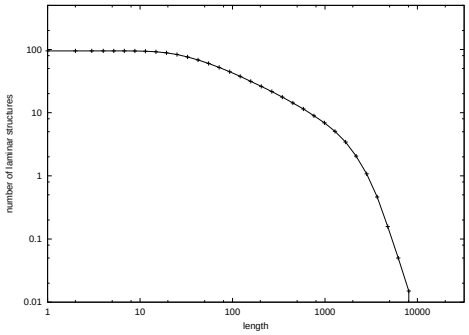


$\epsilon = 1.500,\ \rho = 0.25$

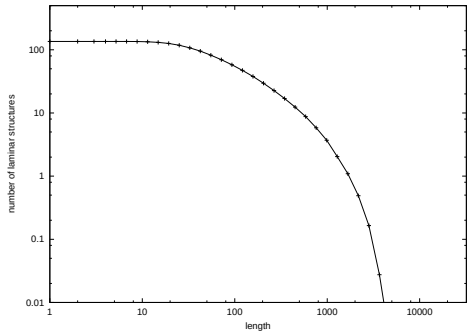
Figure 15.2: continued



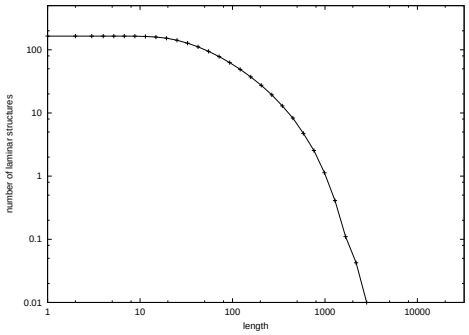
$\epsilon = 1.700,\ \rho = 0.04$



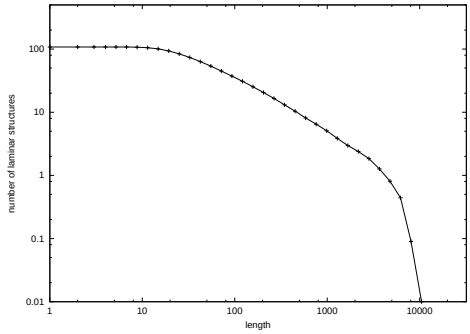
$\epsilon = 1.700,\ \rho = 0.06$



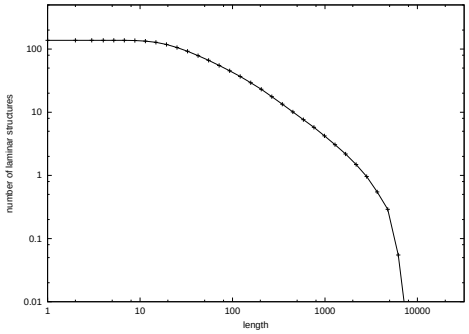
$\epsilon = 1.700,\ \rho = 0.08$



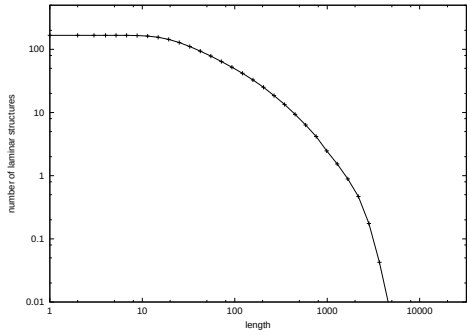
$\epsilon = 1.700,\ \rho = 0.10$



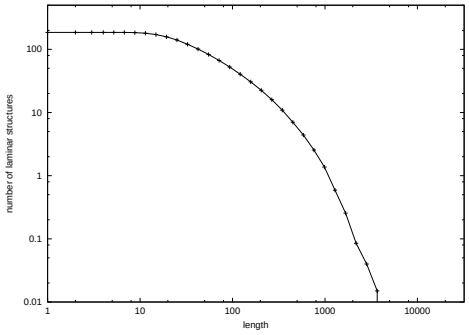
$\epsilon = 1.600,\ \rho = 0.10$



$\epsilon = 1.600,\ \rho = 0.12$

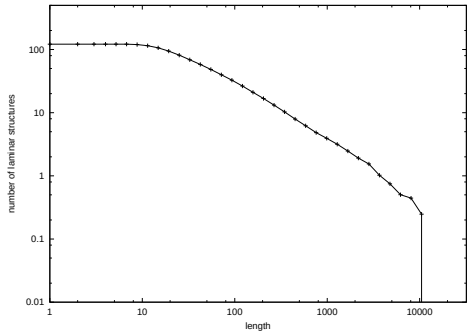


$\epsilon = 1.600,\ \rho = 0.14$

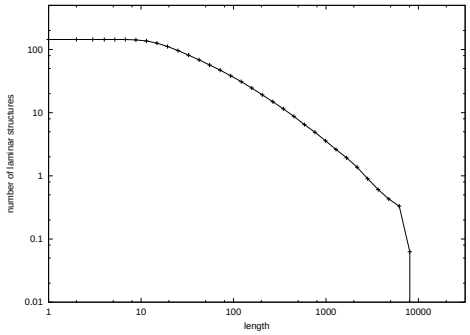


$\epsilon = 1.600,\ \rho = 0.16$

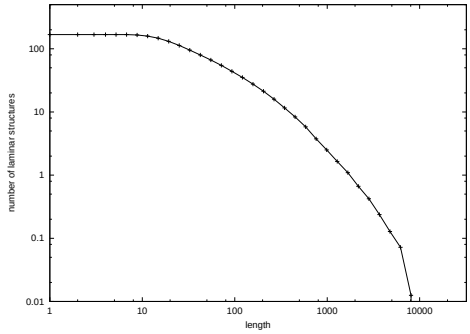
Figure 15.3: number of laminar structure with size greater or equal l



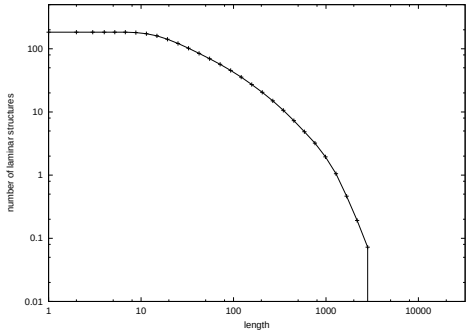
$\epsilon = 1.500,\ \rho = 0.16$



$\epsilon = 1.500,\ \rho = 0.18$



$\epsilon = 1.500,\ \rho = 0.20$



$\epsilon = 1.500,\ \rho = 0.22$

Figure 15.4: continued

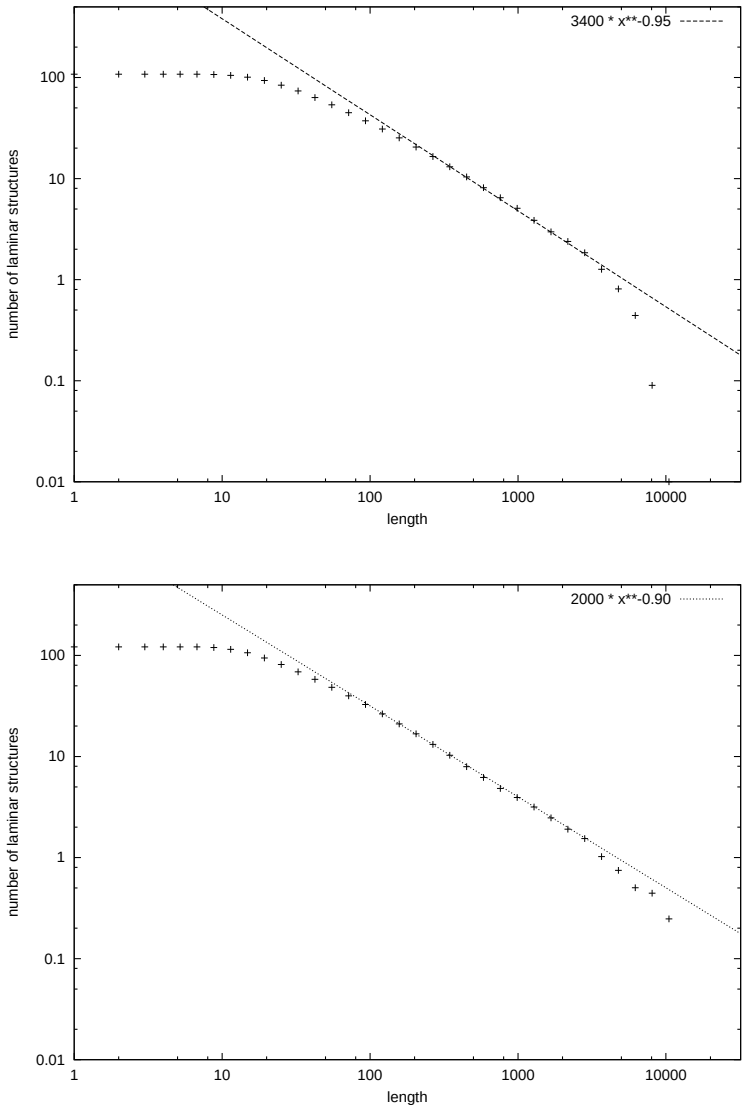


Figure 15.5: ssj

Chapter 16

The slow-to-start model

In the previous chapters, we have analysed the Krauss-model and its similarities with the gas-liquid transition. Compared with other microscopic traffic models, the Krauss-model is relatively complicated. Just understanding what the computation of the safe velocity in (6.2) really does is hard. Then, the model has three parameters (a, b and ϵ) and macroscopic effects always depend on more than one of them. For example, the acceleration depends on both a and ϵ. In this chapter, we apply the same type of analysis done for the Krauss-model to a simpler cellular traffic model.

A *cellular traffic model* is a microscopic traffic model that, apart from time, also discretises space. The road becomes a sequence of cells, eqch of which is either occupied by one car or empty. The speed of a car is the number of cells it advances per time-step, and therefore also discrete. The length of a cell is the space a car occupies in a tight jam (≈ 7.5m), thus the unit of length is the same as in the Krauss-model.

Chapter 17

The Nagel-Schreckenberg model

Probably the simplest and best understood cellular traffic model is the *Nagel-Schreckenberg* (NaSchr) model [3]. Cars can accelerate with one cell per square time-step, i.e. the acceleration rate is one. The braking capability is not limited[1], thus the maximal safe velocity is the number of free cells ahead, i.e. $v_{\text{safe}} = g$. Randomization is done by reducing the desired velocity by one with probability 1/2. The complete algorithm is

$$v_{\text{safe}} = g \tag{17.1}$$

$$v_{\text{des}} = \min(v_{\max}, v_{\text{safe}}, v_f + 1) \tag{17.2}$$

$$r \in_R \{0, 1\} \quad \text{with probability } 1/2 \text{ each} \tag{17.3}$$

$$v_f^+ = \max(0, v_{\text{des}} - r) \tag{17.4}$$

As in the Krauss-model, updates are done in parallel. Unfortunately, the NaSchr-model does not show phase transition [**?**]. A change of the randomization (17.4) suffices to change this.

[1] Due to discretization, the deceleration rate cannot exceed $v_{\max}$

Chapter 18

The VDR-model

To get stable jams between laminar flow, a force is needed that "pulls the jam together". The upstream end of a jam is usually stable without additional forces. On the downstream end, the last car of the jam should be reluctant to accelerate. This is known as *slow-to-start*. It can also be observed in real traffic: When starting after a stop at a traffic light, one tends to leave a longer gap (in seconds, i.e. relative to the speed) than in a dense but easy flowing traffic.

Slow-to-start can be implemented by *velocity dependent randomization* (VDR): the noise term subtracted from the desired velocity depends on the speed of the car. To get slow-to-start, the noise must be higher if the car is stopped. We use

$$r = \begin{cases} \begin{cases} 0 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2 \end{cases} & \text{if } v_f = 0 \\ \begin{cases} 0 & \text{with probability } p \\ 1 & \text{with probability } 1-p \end{cases} & \text{if } v_f > 0 \end{cases}$$

instead of (17.4). p is a parameter. For $p = 1/2$, the NaSchr-model is recovered. For $p \geq 1/2$ the model is no longer slow-to-start. We only consider values for p in $[0, 1/2]$.

This algorithm can be implemented as a lookup-table. The table has an entry for every possible speed v_f and gap g of a car, which gives the speed in the next timestep. Since the model is not deterministic, the entry must list every possible speed v_f^+ along with a probability. Entries for $g > v_{\max}$ are not needed. They are the same as for $g = v_{\max}$, because already there the gap is large enough that the car can drive at full speed. Table 18.1 shows this table for $v_{\max} = 3$.

At this point we want to mention that the parameter ϵ in the Krauss-model also produces a slow-to-start effect. The higher ϵ, the stronger the effect. We have seen this in the discussion of equation (13.2). For $\epsilon \lesssim 0.5$ the slow-to-start effect disappears and the Krauss-model no longer has a phase transition.

	speed							
	0		1		2		3	
0	0		0		0		0	
1	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow 1/2 \\ \leftarrow 1/2 \end{matrix}$	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$
2	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow 1/2 \\ \leftarrow 1/2 \end{matrix}$	$\begin{cases} 1 \\ 2 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 1 \\ 2 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 1 \\ 2 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$
3	$\begin{cases} 0 \\ 1 \end{cases}$	$\begin{matrix} \leftarrow 1/2 \\ \leftarrow 1/2 \end{matrix}$	$\begin{cases} 1 \\ 2 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 2 \\ 3 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$	$\begin{cases} 2 \\ 3 \end{cases}$	$\begin{matrix} \leftarrow p \\ \leftarrow 1-p \end{matrix}$

Table 18.1: Lookup table for the S2S model with $v_{\max} = 3$.

Chapter 19

measurements

If we use $v_{\max} = 3$ as in the Krauss-model, there are only four different possible speeds. For a definition of jams based on a velocity threshold, such a small number of speeds is problematic. We use $v_{\max} = 5$ for all simulations of the S2S model. A jammed car then has speed either zero, one or two.

Figure 19.1 gives some typical space-time plots. $(p = 0.1, \rho = 0.1)$ is in the laminar state, $(p = 0.3, \rho = 0.9)$ is the jammed state. In between – $(p = 0.1, \rho = 0.5)$ and $(p = 0.3, \rho = 0.5)$ – there is phase coexistence. The Nagel-Schreckenberg model at $p = 0.5$ shows the high temperature phase.

Figure 19.2 shows the same systems, but this time immeadeately after they are initialized with the laminar initial condition. Since the space is discrete, it is not possible to put all cars exactly equidistant. At $\rho = 0.5$, one observes coagulation. For $\rho = 0.1$ and $p \leq 0.3$, the cars accelerate without forming a jam.

Finally, Figure 19.3 shows the relaxation after the jammed initial condition. At $p = 0.1$, the jams are stable[1]. At $p = 0.5$, the jams at the downstream end.

Figure 19.4 shows the number of jams as a function of p and ρ. $p \lesssim 0.2$ produces one stable, big jam. For $p \gtrsim 0.2$ the jam becomes instable and breaks into multiple smaller jams. At high densities, no car can drive fast, thus a single jam covers the entire road. The higher the noise, the less density is necessary to jam the entire road. Finally, we find a laminar structure and no jams for $p < 0.2$ and $\rho < 0.1$.

The density variance is shown in Figure **??**. For $\rho < 0.2$ and medium densities, traffic is very inhomogeneous due to phase coexistence. In contrast to the Krauss-model, the laminar and the jammed state are almost not present. For $\rho \to 0$, there cannot be a jam. But already at $\rho = 0.1$ jams begin to form. Similarly, for $\rho \to 1$, no car can be in laminar flow, but fast cars appear when ρ is only slightly less than one. The jams thus must be very dense. In the Krauss model each car in a jam can leave a small gap to the next car, resulting in jams with densities . In the S2S model, cars cannot leave small gaps, they are either tight together or have a distance of one. It seems that such empty cells in a jam tend to merge and form small laminar structures.

For $p > 0.2$, traffic becomes more and more homogeneous at all densities. The Nagel-Schreckenberg model at $p = 0.5$ is almost completely homogeneous.

[1] the fact that for $\rho = 0.1$ the jam disappears does not mean that a jam is in principle instable

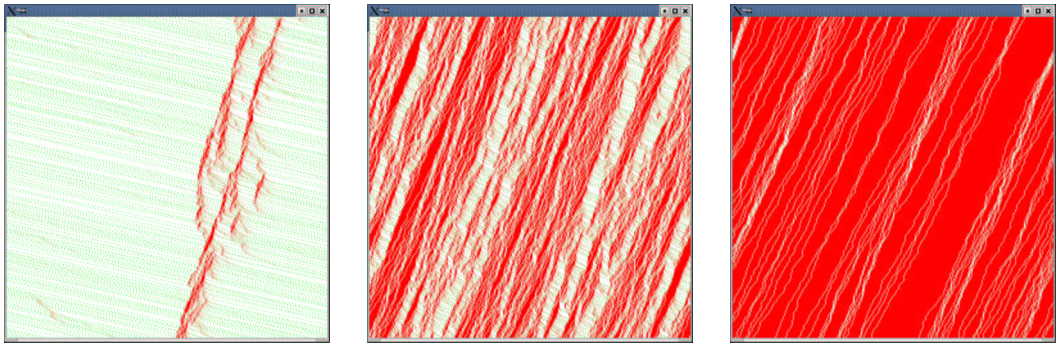
It has only a single phase, namely the high temperature phase. The isolines at $p > 0.2$ are slightly elliptic with a diagonal axis, similar to the Krauss-model, but far less . For the Krauss-model we explained this by the dependence of the acceleration (and thus of the density of the laminar structure) on ϵ. In the S2S model, p does not influence the acceleration, because for a stopped car the noise is locked to $1/2$.

Figure 19.6 top showing the density of jams confirms what was said in the discussion of Figure 19.5. For $p < 0.2$, that density of the jams is very high (> 0.9). Further, in the entire coexistence phase this density is independent of the global density ρ. This is also true for the density of the laminar structures (Figure 19.6 bottom). That density is also fairly independent of p.

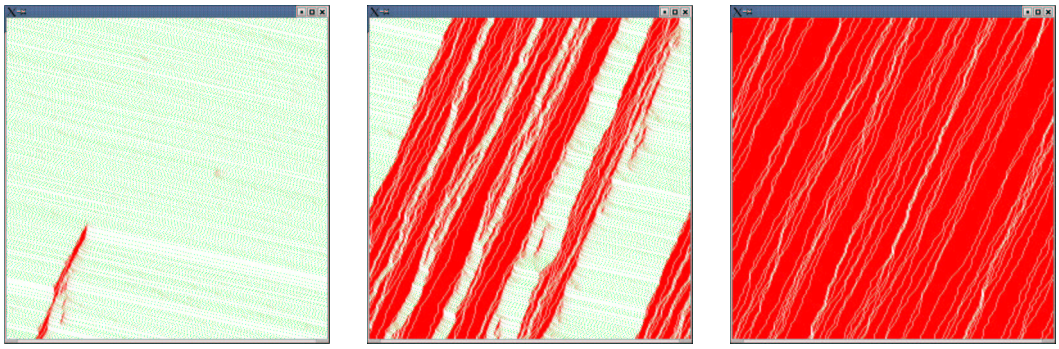
Now have a look at the distribution of the lengths of the laminar structures (Figure **??**. $(p = 0.1, \rho = 0.1)$ is the laminar state. A single laminar structure covers the entire road. $(p = 0.1, \rho = 0.5)$ and $(p = 0.1, \rho = 0.9)$ show the coexistence state. There is always a laminar structure of length about $L - N$. The small laminar structures are the result of a not perfectly stable interface at the downstream end of the jams. At $(p = 0.5, \rho = 0.9)$ the system is completely jammed. At $(p = 0.3, \rho = 0.9)$ the road is still jammed, but there are a few faster cars.

All measurements indicate that the S2S model has a two phase regime for low noise values and a one phase regime at high p. Compared to the Krauss-model, there are three differences:

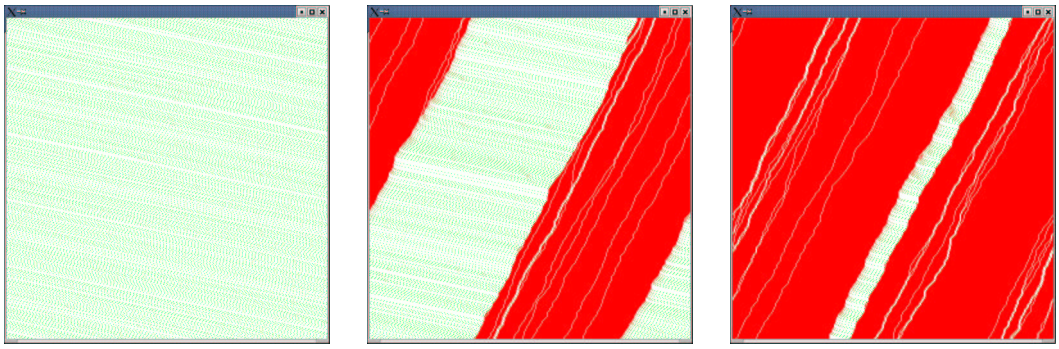
- The Krauss-model has a second single phase regime for $\epsilon < 0.5$. In the S2S model, the two phases remain when $p \to 0$.
- The laminar state and the jammed state almost disappear for small p values, there is mostly phase coexistence. The Krauss model in the two phase regime, shows the laminar and the jammed state over a relatively large intervals of ρ values. This is because there is a significant amount of noise in the entire two phase regime. For $\epsilon \to 0$, the Krauss-model is in the single phase regime and there are anyway no jams or laminar structures.
- As already explained, the coexistence curve of the Krauss-model "leans" far more to the left.



$p = 0.5,\ \rho = 0.1$ $p = 0.5,\ \rho = 0.5$ $p = 0.5,\ \rho = 0.9$

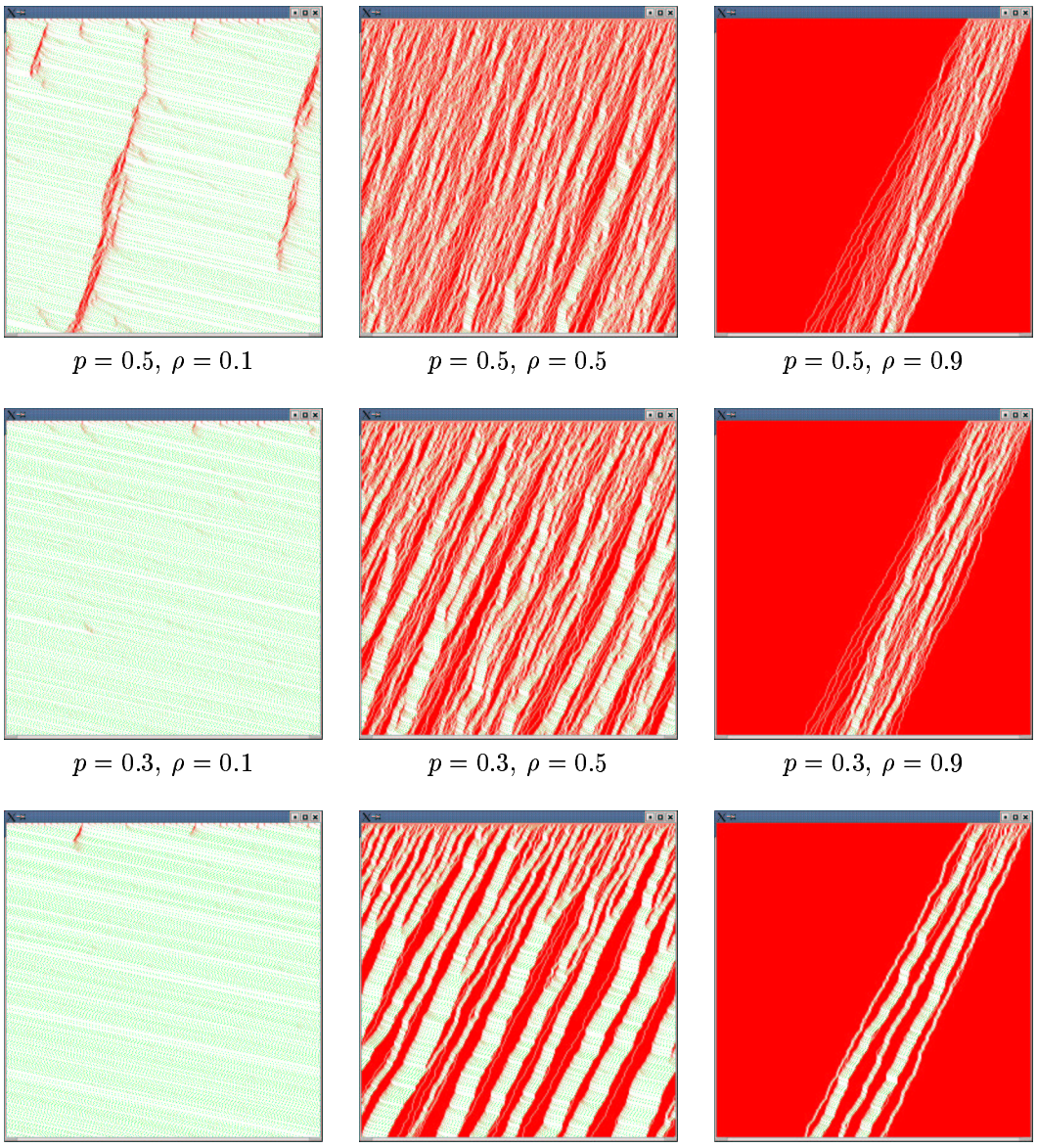


$p = 0.3,\ \rho = 0.1$ $p = 0.3,\ \rho = 0.5$ $p = 0.3,\ \rho = 0.9$



$p = 0.1,\ \rho = 0.1$ $p = 0.1,\ \rho = 0.5$ $p = 0.1,\ \rho = 0.9$

Figure 19.1: Space-time plots for different parameters of the S2S model. Space is horizontal; time increases downward; each line is a snapshot; vehicles move from left to right; fast cars are green, slow cars red. Displayed are 500 time-steps for a road of length $L = 500$. The systems have run for some time before the pictures were taken, they are in equilibrium.



$p = 0.5,\ \rho = 0.1$ $p = 0.5,\ \rho = 0.5$ $p = 0.5,\ \rho = 0.9$

$p = 0.3,\ \rho = 0.1$ $p = 0.3,\ \rho = 0.5$ $p = 0.3,\ \rho = 0.9$

$p = 0.1,\ \rho = 0.1$ $p = 0.1,\ \rho = 0.5$ $p = 0.1,\ \rho = 0.9$

Figure 19.2: Space-time plots for systems started with the laminar initial condition. Otherwise the systems are equal to those of Fig. 19.1.

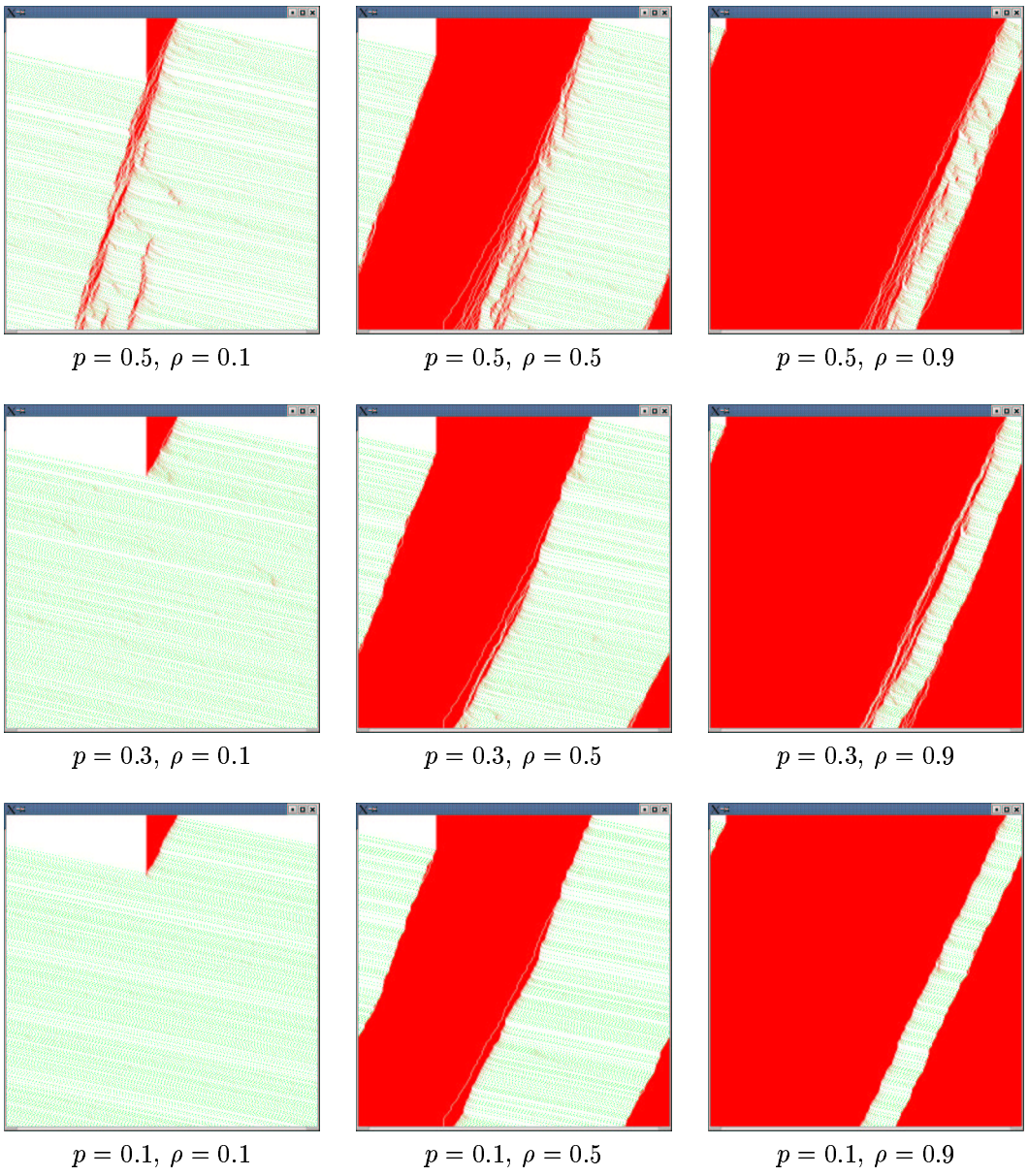


Figure 19.3: Space-time plots for systems started with the jammed initial condition. Otherwise the systems are equal to those of Fig. 19.1.

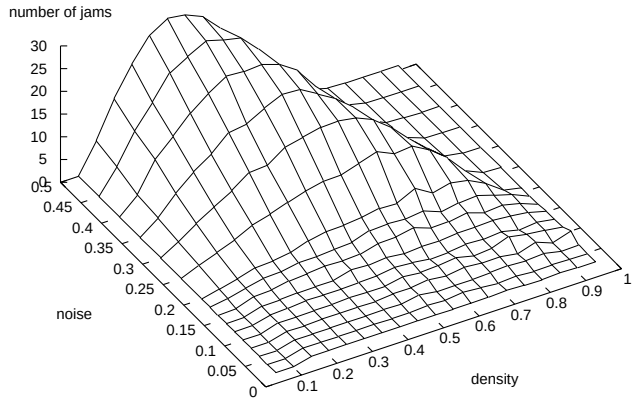


Figure 19.4: Number of jams as a of the density ρ and the noise p. $L = 4000$

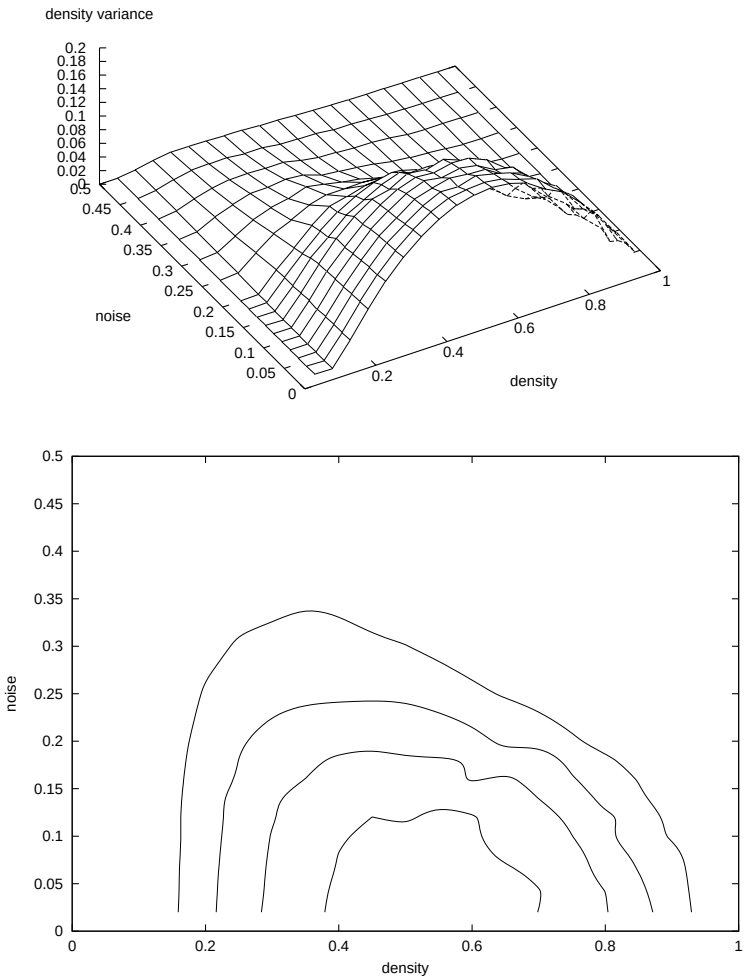


Figure 19.5: 3d-plot and isolines of the density variance. The outermost isoline is $\mathrm{Var}[\rho_l] = 0.01$, the innermost $\mathrm{Var}[\rho_l] = 0.09$. $L = 4000$ and $l = 62.5$

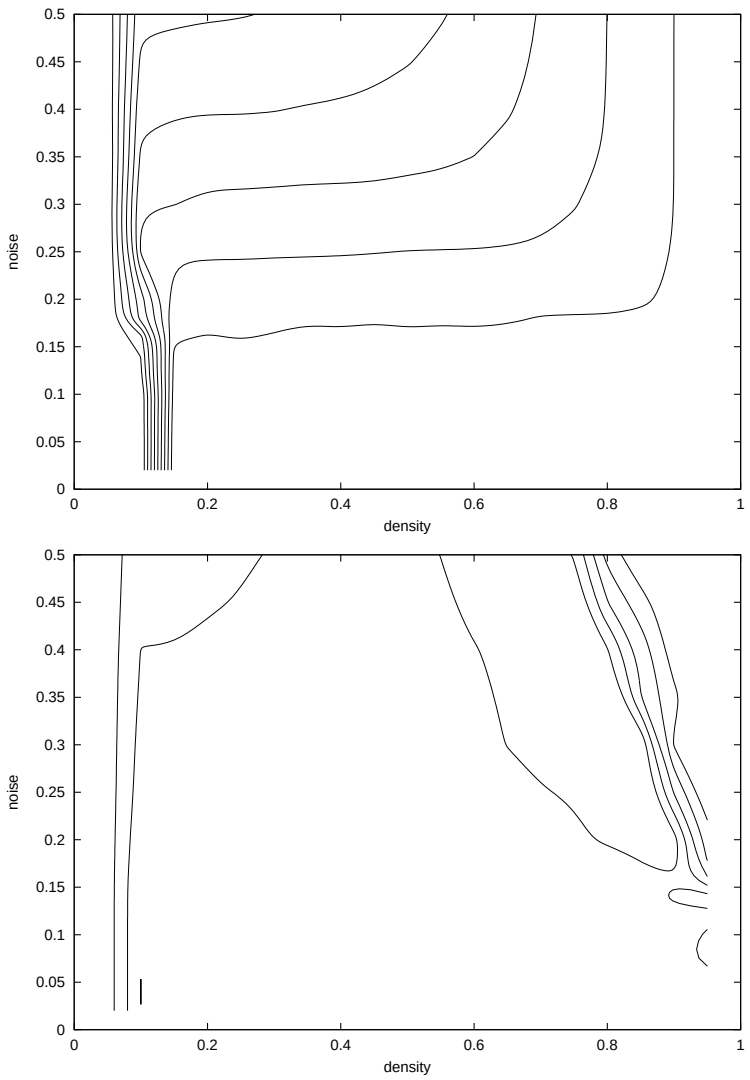
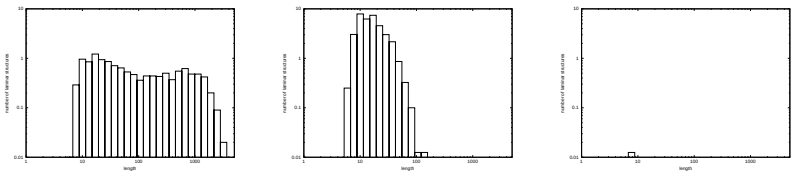
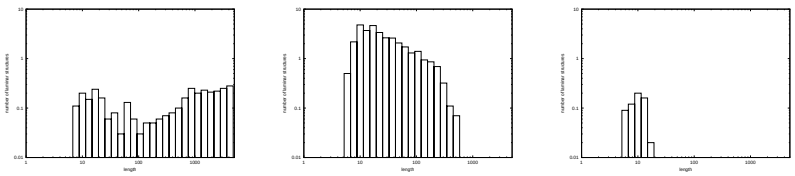


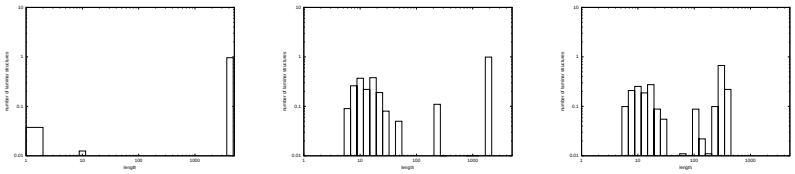
Figure 19.6: TOP: Isolines for the average density of a jam. Leftmost isoline is 0.1, rightmost 0.9. $L = 4000$. BOTTOM: Isolines for the average density of a laminar structure. Topmost isoline is 0.03, bottommost 0.21 (in steps of 0.03). $L = 4000$



$\epsilon = 0.500,\ \rho = 0.10$ $\epsilon = 0.500,\ \rho = 0.50$ $\epsilon = 0.500,\ \rho = 0.90$



$\epsilon = 0.300,\ \rho = 0.10$ $\epsilon = 0.300,\ \rho = 0.50$ $\epsilon = 0.300,\ \rho = 0.90$



$\epsilon = 0.100,\ \rho = 0.10$ $\epsilon = 0.100,\ \rho = 0.50$ $\epsilon = 0.100,\ \rho = 0.90$

Figure 19.7: Histograms of the lengths of the laminar structures. $L = 4000$ for all plots. The heights of a bar is the expected number of laminar structures with length in the interval corresponding to the width of the bar. Plots are in log-log-scale.

Chapter 20

Computational issues

A standart Pentium III 700MHz processor can compute about 2.5 million car updates per second. Whether the Krauss-model or S2S is simulated does not change much. A system with 1000 cars takes about a million time steps to reach equilibrium. That value depends on the other parameters. Larger systems take longer to relax. In order to get good (averaged) data, one needs to simulate about 10 systems. Remember that you also average over time, as explained in section 10. Thus, getting the data for one point in the noise/density plane takes about one hour. The plot in Figure 13.1 contains about 400 points, which would take about two weeks to compute on a single processor. The good news is that different points in the noise/density plane can be computed in parallel on multiple CPUs.

Chapter 21

Discussion

There is no general agreement if measurements show 1-phase/1-state or 2-phase/3-state traffic (or possibly even three phases [5]). There is some evidence for hysteresis in Germany [4], manifesting itself in transitions from high to lower flow values at the same density. Hysteresis, which was also found earlier [8], is a strong indication for a 2-phase model. However, even in Germany, most measurements indicate highly variable traffic at intermediate densities, which does not correspond to any clear-cut picture.

In this context, one should note that a 1-phase model which is close to a 2-phase model would also display highly variable traffic at intermediate densities, although it would be homogeneous at large scales as discussed in Sec. **??**. This variability is however a property of stochastic models only and for that reason it is not well integrated into current theory development. A precise investigation of these relations is beyond the scope of this paper. It seems however impossible to us to clarify the question if traffic displays several phases or not – and therefore, if breakdown probability should be entered into the Highway Capacity Manual or not – without having understood how different phases are generated by stochastic models. The present work fills exactly this gap.

Chapter 22

Summary

This paper shows, via numerical evidence, that a specific stochastic car following model can either display 1-phase/1-state or 2-phase/3-state traffic, depending on the noise parameter. With 2-phase parameters, the two phases are laminar and jammed, which also corresponds to two of the three states. Those states are homogeneous. The third state, at intermediate densities, is a coexistence or mixed state, consisting of sections with jammed and sections with laminar traffic.

The transition to a 1-phase/1-state model happens via the densities of the laminar and of the jammed phase approaching each other until they become the same. Beyond this point, there is only one homogeneous phase of traffic. In the gas-liquid model, this point is a critical point. Spatial correlations grow to infinity and the system can become fractale. Whether the Krauss-model also has a critical point could not be answered.

In our view, it is important to understand this possibility of stochastic models to be in different regimes if one considers to enter discussions of traffic breakdown probabilities into the Highway Capacity Manual. If traffic is best described by a 1-phase model, then there is, in our view, no theoretical justification for such probabilities. If, however, traffic is best described by a 2-phase model, then the 2-phase model could even give theoretical predictions for breakdown probabilities. A discussion of breakdown probabilities in 2-phase models can be found in Ref. [**?**].

Acknowledgments

We thank F. Wagner for intensive discussions. Also D. Stauffer and M. Droz have significantly helped in defining good computational criteria for the detection of multiple phases. P. Wagner and C. Daganzo have helped indirectly (and maybe inadvertently), via the discussion of a separate but related paper. A significant portion of the computing times on 192-CPU cluster Xibalba and on 44-CPU cluster Linneus was used for some of the computational results.

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