Dynamic market simulations

–diploma thesis–

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Contents

Chapter 1

Overview

General economic equilibrium theory is static. Since there is no dynamics, little is known how the system should reach the equilibrium state, or how it behaves once it is away from it. Actually, the situation is somewhat similar to the situation in Statistical Physics some 100 years ago, where the macroscopic thermodynamic equations were around, but neither was their microcsopic foundation nor their extension into non-equilibrium.

In this thesis, we present a *dynamic* model of an agent-based (in the language of Statistical Physics "atomic") economy. We construct the dynamics so that the economy evolves towards the general equilibrium result when the simulation is started in non-equilibrium. Since the model is dynamic, it is not possible that the system will reach the exact equilibrium point, as there is always some kind of noise in the system that disturbs the evolution process. On the other hand, the equilibrium point is also not known exactly, as the analytical equations cannot be solved. It will always be possible, though, to find an approximate analytical solution for the equilibrium point which can be compared to the simulation result.

Generally, the equilibrium point of an economic system is the situation where no agent can increase her utility by a change of her strategy. In the language of evolutionary game theory, it is the Nash equilibrium. In our model economy, there is only one kind of commodity which the agents can produce, sell and consume. This commodity is not quantized, i.e. the agents can manipulate any quantity of it. The *strategy* of an agent then specifies her consumption behaviour as well as the price that she charges for the selling of the commodity. The *dynamics* of the model specifies how the agents can adapt their strategies, i.e. it specifies on which timescales an by how much the individual economic quantities that form the strategy can be changed. Of course, the force which drives the adaption is the goal to find a strategy that optimizes the utility.

As usual, the utility function of the agents consists of two terms: One term describes the *dis*utility related to the work the agent has to perform; the second term describes the gain in utility that comes from consumption. We consider two kinds of consumption utilities which lead to very different behaviours of the agents in the limit of systems with many agents:

In chapter 2 we consider "square root consumption utility". Here, the agents are very anxious to spend their money in a way that considers every supplier: Buying nothing from an agent, even if the price is very high, results in a great loss of utility. In contrast, in chapter 3 where the consumption behaviour of the agents is governed by "logistic consumption utility," substitution is always at hand, especially in the limit of many agents. If the price of an agent is much above average then, she will not sell anything at all, as all agents decide to buy more of the other (less expensive) products instead. This results in very large fluctuations when the agents adapt prices, and it is much more difficult to choose dynamics that leave the system stable and give reasonable equilibrium results. In order to be able to compare the results of the two models we tried, though, to make as little changes in the dynamics as possible when we switched from square root to logistic consumption utility.

Chapter 2

The basic model: square root consumption utility

2.1 Model description

The basic model consists of N agents, each of them producing exactly one good. Each agent is characterized by a utility function

$$U_i = -\frac{q_i^2}{2} + 2 \sum_{j \neq i} \sqrt{x_{ij}} \; . \tag{2.1}$$

q_i is the amount of work of agent i; x_{ij} is the amount of good j (bought from agent j) which agent i consumes. The conversion of work into (dis)utility is convex, meaning that working twice as much is more than twice as unpleasant. In contrast, consumption is concave, meaning that consuming twice as much is less than twice as pleasant.[1]

The model works as follows.

Initialization:

- Each agent i receives an initial amount of money $M_{i,0}$.
- Each agent i sets an initial price $p_{i,0}$.
- Each agent i sets, for each good j, an initial demand $\hat{x}_{ij,0}$. These demands set the ratios between the goods, that is, we always use $\sum_j \hat{x}_{ij} = 1$.

One time step:

1. Agents order goods according to their demands vector $\hat{x}_{ij}$ such that exactly all money is spent. This means a rescaling of the $\hat{x}_{ij}$:

$$x_{ij} = \hat{x}_{ij} \, \frac{M_{i,t-1}}{\sum_{j \neq i} \hat{x}_{ij} \, p_j} \; . \tag{2.3}$$

[1]In this paper, Eq. 2.1 will be used. However, it can be seen as a specific version of

$$U_i = -A_i \, g(q_i) + \sum_{j \neq i} a_{ij} h(x_{ij}) \; , \tag{2.2}$$

where $g(X)$ is a convex function and $h(X)$ a concave function. Many of our results should also hold for this more general version of the utility function.

$M_{i,t-1}$ is the amount of money left over from the last time step, see below. Note that now $\sum_{j \neq i} x_{ij}\, p_j = M_{i,t-1}$, which is the **budget constraint**.

2. Goods are produced to order, and sold at the prices previously indicated. That is, each agent produces

$$q_i = \sum_{j \neq i} x_{ji} \tag{2.4}$$

and receives the amount of

$$M_{i,t} = p_i\, q_i \; . \tag{2.5}$$

Since, according to Eq. 2.3 each agent has spent all her money with her orders, the result of Eq. 2.5 is indeed the amount of money with which they go into the next time step.

3. Utilities U_i are calculated.

4. Adaptation of demands: Every T_x time steps, agents adapt (see Sec. 2.2) their x_{ij} to maximize utility given prices. Usually, $T_x = 1$.

5. Adaptation of prices: Every T_p time steps, agents adapt (see Sec. 2.2) their p_i to maximize utility. Adaptation of prices happens on a much slower time scale than adaptation of demands, i.e. $T_p \gg T_x$. Usually in this paper, $T_p = 10^5$.

In an alternative version we replace points 4. and 5. by a different adaption scheme: We allow the agents to directly *calculate* the optimal demands given prices. This reduces T_p dramatically:

4. Adaption of demands: Every T_x time steps, agents calculate (see Sec. 2.4.1) their optimal x_{ij} to maximize utility given prices.Usually, $T_x = 1$.

5. Adaptation of prices: Every T_p time steps, agents adapt (see Sec. 2.2) their p_i to maximize utility. Adaptation of prices happens on a slower time scale than adaptation of demands: $T_p = 10$.

2.2 Adaptation

The above model description does not specify how adaptation takes place. In fact, many different adaptation schemes work. Throughout this paper, we use "trial-and-error" adaptation. This means that, from time to time, agents try different strategies. If the performance (= utility) of a changed strategy turns out to be better than the performance of the previous strategy, then they will stick with the new strategy. For each agent i, a strategy consists of the $(\hat{x}_{ij})_j$ and of p_i. Trial-and-error adaptation for the $\hat{x}_{ij}$ works as follows:

1. After T_x time steps with the "normal" strategy, the agent picks randomly a j between 1 and N and thus the corresponding strategy entry $\hat{x}_{ij}$. The agent remembers its old value, plus the corresponding utility U_i, and then generates a new value for $\hat{x}_{ij}$.[2]

2. The agent operates with the new trial value for T_x time steps and then looks at the resulting utility. If this new utility is larger than the original one, she sticks with the new $\hat{x}_{ij}$, otherwise she returns to the previous demands vector.

[2] According to the above model specification, this means that all $\hat{x}_{ij}$ will be rescaled such that still $\sum x_{ij} = 1$.

Trial values can be obtained via two mechanisms:

- **Mutation**. The old x_{ij} is changed by a small random amount.
- **Copy**. A new value of x_{ij} is taken from another agent.

For this paper, only mutation will be used.

The above explanation was in terms of demands $\hat{x}_{ij}$. Adaptation for p_i happens accordingly, the three main differences being that (1) there is only one price per agent, (2) the trial time T_x is replaced by T_p, and (3) agents enter into trial mode only with probability p_{trial} where we use $p_{trial} = 0.1$.

More parameters are necessary to fully describe adaptation; for example, a mutation for $\hat{x}_{ij}$ is, with probability $1/2$, either $\hat{x}_{ij,new} = \hat{x}_{ij,old} \times f_{x,rand}$ or $\hat{x}_{ij,new} = \hat{x}_{ij,old}/f_{x,rand}$ where $f_{x,rand}$ is a random number between 1 and 1.01. For the price adaption we mutate accordingly with a factor $f_{p,rand}$ which is a random number between 1 and 1.1. Our simulations indicate that our results are robust as long as the mutations remain small, and as long as T_p remains long enough so that the adaptation of demands can complete even with small mutations.

2.3 Related work

The model is related to a model by Bak, Norrelyke, and Shubik (BNS) [1]. The main difference is that in the present paper, agents transparently adapt x_{ij} (via $\hat{x}_{ij}$) and p_i, which seem to be the plausible economic quantities to work with. In addition, the present paper assumes that everybody is buying from everybody, whereas the BNS model assumes that everybody is buying from their "left" neighbor only. No differences between this model and the BNS model will be analyzed here, but for us, the work of BNS provided a good first example of a simulation of an agent-based economy.

2.4 Analytical approximation

The above model corresponds to a solution of a two-step optimization problem:

1. First, for given prices $(p_j)_j$ and money $M_{i,t-1}$ find an optimal allocation of the $\hat{x}_{ij}$ such that utility is maximized. This is done in Sec. 2.4.1.
2. Second, given that every agent knows every other agent's reaction to price changes, find an optimal price. See Sec. 2.4.2.

2.4.1 Optimal consumption

For given prices and money, the utility function of every agent is a sole function of the demands x_{ij}:

$$U_i := -q_i^2/2 + 2\sum_{j\neq i} \sqrt{x_{ij}} + \lambda_i \left(M_{i,t-1} - \sum p_j x_{ij} \right) . \qquad (2.6)$$

$M_{i,t-1}$ is the money from the last time step; the Lagrangian multiplier means that all money has to be spent.[3]

Now consider a fixed agent i. Since $q_i = \sum_{j\neq i} x_{ji}$ and $M_{i,t-1}$ do not depend on x_{ij}, the derivative with respect to x_{ij} is simple, and the derivative with respect to λ_i is just the budget constraint. The roots of the two expressions are

$$x_{ij} = \frac{1}{\lambda_i^2\, p_j^2} \qquad \text{and} \qquad \frac{1}{\lambda_i^2} = \frac{M_{i,t-1}}{\sum_{k\neq i} \frac{1}{p_k}}. \tag{2.7}$$

The resulting full expression for the optimal consumption given prices and money is therefore

$$x_{ij} = \frac{M_{i,t-1}}{p_j^2 \sum_{k\neq i} \frac{1}{p_k}}. \tag{2.8}$$

Note that for a homogeneous solution, we can drop all indices and obtain the obvious result $x = \frac{M/p}{N-1}$ which must be true for any utility function.

2.4.2 Optimal price

It is much harder to find an expression for the optimal price which an agent should choose if all other prices and the reaction Eq. 2.7 is known. The relevant pieces of the utility function are

$$\tilde{U}_i = -\frac{1}{2}\left(\sum_j \frac{1}{p_i^2\,\lambda_j^2}\right)^2 + \lambda_i\, p_i \sum_j \frac{1}{p_i^2\,\lambda_j^2} \tag{2.9}$$

$$= -\frac{1}{2}\, p_i^{-4}\left(\sum_j \frac{1}{\lambda_j^2}\right)^2 + \lambda_i\, p_i^{-1} \sum_j \frac{1}{\lambda_j^2} \tag{2.10}$$

Note that in contrast to the previous section where the “pleasure” of consumption was balanced against the “pain” of paying for it we balance now the displeasure of work against the pleasure of getting money for it.

Large N limit

In the limit $N \to \infty$, the λ_j become independent of p_i since the contributions of p_i to them is of order $1/N$. λ_i does not depend on p_i anyway. Maximization of the above expression for the utility w.r.t. p_i and then passing to the homogenous case where $\lambda_i^2 = \lambda_j^2 = \lambda^2 = (N-1)/Mp$ gives the result

$$p \approx \frac{2^{2/3}\, M}{(N-1)^{1/3}}. \tag{2.11}$$

[3] This is already an approximation. Since the model is stochastic, income will not always be the same. However, in the current formulation, the x_{ij} are slowly varying variables. That is, one would have to maximize the expectation value of U_i over the distribution of M_{t-1}.

A cautionary remark about the interpretation of this formula should be made here, because we did two important approximations: First we only consider the "large N" limit, secondly the calculation does not include the fluctuations that are caused by the testing of new prices. Now note that M is the average amount of money per agent, but more exactly it is M_{SS} (the average amount of money which the agents that are in the *steady state* have) which should be used. The point is that it is possible that $M \neq M_{SS}$ as a consequence of the fluctuations in money due to the constant testing of new prices in equilibrium: If, in equilibrium, an agent tests a lower price, her money will fluctuate upwards and if she test a higher price, it will fluctuate downwards. Like this, $M \neq M_{SS}$ if the agents that currently are not testing (the ones that are in the steady state) do not give as much more money to the agents that test lower prices than they give less to the agents that test higher prices. This model with square root consumption utility does not (to leading order) show such an asymmetry, so that we can safely set M_{SS} equal to $M = M_{tot}/N \equiv 1$. In a model where it is easy to substitute one product for another (e.g. the model with a logistic utility function, see chapter 3) agents that test high prices will lose all their money, while the ones that test low prices will earn very much. In this case the average money of the equilibrated agents will be less than M_{tot}/N and fluctuations must be included in the calculation of the Nash equilibrium price.
Now inserting Eq. 2.11 back into Eq. 2.7 gives the relaxed state demands, production and thus also utility level as a function of N alone:

$$x \approx \frac{2^{-2/3}}{(N-1)^{2/3}} \tag{2.12}$$

$$q = (N-1)\,x \approx 2^{-2/3}\,(N-1)^{1/3} \approx 0.63\,(N-1)^{1/3}\;, \tag{2.13}$$

and

$$U \approx (2^{2/3} - 2^{-7/3})(N-1)^{2/3} \approx 1.388 * (N-1)^{2/3}. \tag{2.14}$$

That is:

- Prices are proportional to the amount of money M in the system.
- Prices decrease with increasing N.
- Consumption does not depend on prices.
- The consumption of each individual good decreases with increasing N. However, the sum of all consumption, $\sum_j x_{ij}$, which is just the production in the homogenous case, *in*creases with N. This is a consequence of fact that the slope of the utility function goes to infinity as $x \searrow 0$, meaning that an ϵ amount of each good gives an infinite increase in utility[4].

It is important to note here that we are looking for the *Nash* equilibrium price level and not the system optimum: If all agents would work together they could agree upon a price p which would maximize the utility of all of them at the same time. This price can be calculated by inserting the "homogenous

[4]Note that this is quite a unrealistic behaviour, as one expects the production (as well as utility) to be bound for $N \to \infty$ (likewise one would expect a lower bound $p_\infty > 0$ for the price in this limit). In chapter 3 where the agents show logistic consumption behaviour, this will indeed be the case, but the price we have to pay to get that result will be that the model shows very large (and likewise unrealistic) fluctuations.

budget constraint" $x = \frac{M/p}{N-1}$ into the utility function and maximizing the resulting function w.r.t. p. It would be lower than the Nash equilibrium price:

$$p_{SO} = \frac{M}{(N-1)^{1/3}}. \tag{2.15}$$

and (by reinsertion into U) give the agents the maximum utility

$$U_{SO} = \frac{3}{2}(N-1)^{2/3}. \tag{2.16}$$

The situation is the same as in the "prisoners dilemma" where both players would be better off if they played together, but because every player tries to maximize his own profit, both are worse off in the end. The mechanism of trying to maximize only the own utility makes the system optimum unstable: In the present case, if the system is in the system optimum, one agent can profit by charging a higher price, but because all agents try to do the same, the system evolves towards a lower average utility level. It is the Nash equilibrium, therefore, which is an attractor of the dynamics, and in order to find the equilibrium price level one always has to consider only one agent trying to find the optimal price for herself and only then consider the homogenous case.

Strictly speaking, in order to find an expression for the optimal price of an agent i in the nonhomogenous case, it is not enough to consider only two timesteps (i.e. to consider only the reaction Eq. 2.7 that the agents will show in the timestep after the price change), because the redistribution of money after a price change takes longer. After a price change several timesteps are needed until a new steady distribution of money is established (see Sec. 2.9). The steady state condition

$$M_i = p_i \sum_{j \neq i} x_{ji}, \tag{2.17}$$

(with x_{ji} given by Eq. 2.8) yields (see Sec. A.1.1 in the appendix) that if the prices are fixed the distribution of money will converge to

$$M_i = \frac{M_{tot}}{2} \frac{\sum\limits_{j_1<j_2<\ldots<j_{N-2}, j_k \neq i} p_{j_1} p_{j_2} \ldots p_{j_{N-2}}}{\sum\limits_{j_1<j_2<\ldots<j_{N-2}} p_{j_1} p_{j_2} \ldots p_{j_{N-2}}}. \tag{2.18}$$

The sums go over all $(N-2)$-tuples $(j_1, j_2, \ldots, j_{N-2})$ of integers between 1 and N. All $j_k, k = 1, \ldots, (N-2)$ must be different and the order is not relevant. For the sum in the numerator, all j_k must be different from i. This sum has only $N-1$ summands.
If this is known the resulting production and demands of all agents are known as well (by use of Eq. **??**) and therefore also the resulting utility, which becomes a sole function of the prices and M_{tot}. Maximization of the utility with respect to p_i can then be performed in principle, but the resulting equations are too complicated to treat analytically.

In Sec. A.1.1 a further analysis of Eq. 2.18 is performed for the case where all agents except agent i charge the same price p. This analysis yields for example that if $p = p_{SO}$, agent i should optimally charge a lower price than p_{SO}, which shows that the system optimum is unstable (as stated above).

2.5 Simulation results

2.5.1 Adaptation of demands

First, we check if the simulation reacts to price differentials. For that, we set one agent at a different price than all others (Figs. 2.1 and 2.2). One sees that agents slowly shift their consumption towards the good with the lower price, visible in the graphs as a larger production for that agent. One also sees that this results in a higher utility for that agent, meaning that in this case average prices are too high and an individual has an advantage when charging a lower price. This effect is what will drive the adaptation of prices in Sec. 2.5.2.

The validity of the analytical solution (i.e. Eq. 2.18) can be verified here. In the case where all agents except one have the same price, the formula reduces to

$$M_i = M_{tot} \frac{p}{2p + (N-2)p_i} \tag{2.19}$$

for agent i with price p_i and

$$M \equiv \frac{M_{tot} - M_i}{N-1} = \frac{M_{tot}}{N-1} \frac{p + (N-2)p_i}{2p + (N-2)p_i} \tag{2.20}$$

for the rest. These equations must be exactly valid for all numbers of agents, as we did no approximations. Because the result for the optimal demands given prices (Eqs. 2.7) was used for the derivation of Eq. 2.18, these formula is also implicitly verified. For the simulation where $M_{tot} = N$, $p = 3$ and $p_i = 2.4$ were used, the above expressions evaluate to

$$M_i = 1.19047 \tag{2.21}$$

for $N = 10$ and

$$M_i = 1.23762 \tag{2.22}$$

for $N = 50$. The simulations were run for $2 * 10^6$ timesteps and, indeed, an exact equality of the mean allocation of money in equilibrium and the above values is found.

One sees in Figs. 2.1 and 2.2 that it takes about 40000 timesteps until the aggregate consumption has relaxed to the value corresponding to the prices. Including some "safety margin," we will use $T_p = 100\,000$ in the following simulations.

Some cautionary remarks about this choice of $T_p = 100\,000$: First of all, this figure would need adjustment if we'd want to simulate an economy with more agents: Since every agent can adapt only one x_{ij} per timestep, T_p scales linearly with N. Secondly one has to be aware that the system considered in this section is far away from equilibrium. Indeed, according to Eq. 2.11 the steady state price is approximately 0.8 in the system of 10 agents and approximately 0.4 in the system with 50 agents. If the system is that much away from equilibrium, there is a strong signal for the agents that they should lower the price. This signal gets weaker on the way to equilibrium and thus adaptation time for the demands will get longer, demanding a higher T_p. Thus, the inclusion of the above mentioned "safety margin" cannot be avoided. This subject will be discussed further in Sec. 2.10.

2.5.2 Adaptation of both demands and prices

We now allow both the demands $\hat{x}_{ij}$ and the prices p_i to adapt. As explained above, the adaptation of the $\hat{x}_{ij}$ happens on a fast time scale, while adaptation of the p_i happens on a slow time scale. In practice, this means that every other time step every agent tries a new $\hat{x}_{ij}$[5]. In contrast, new prices are tried out only every $T_p = 10^5$ time steps, and they are left in place for the same duration before they are evaluated.

As a result, this simulation takes much longer, note the time scale in Figs. 2.3. One also sees how the system relaxes towards a price of about 0.87, a production of about 1.2, and a utility of about 5.7.

For comparison, the "large N" solution from Sec. 2.4.2 gives, for $N = 10$ and $M = 1$ used here, $p \approx 0.76$, $q \approx 1.31$ and $U \approx 6.01$. It is clear that the values of the simulation will not completely agree with the values of the analytical calculation since the assumptions are different. For example, the analytical calculation is valid only for large N (compared to $N = 10$ in the simulation), and the analytical calculation does not take into account fluctuations.

Fig. 2.4 shows the results for the simulation with 50 agents. Since in this model every agent can buy from every other agent, computation time always scales with the square of N. Thus, simulations with that many agents take a long time. In this case, the simulation needs several days of computation time. The tabular below summarizes the simulation results of this section:

N=10	price	production	utility
simulation	0.87	1.2	5.7
analytical	0.76	1.3	6.0
rel. error	0.13	0.08	0.05

N=50	price	production	utility
simulation	0.461	2.16	18.22
analytical	0.434	2.31	18.59
rel. error	0.06	0.07	0.02

(2.23)

Because the analytical solution is only valid for large N, one expects the simulation results closer to the analytical solution in the simulation with 50 agents. This is indeed verified as the relative error decreases. As explained in Sec. 2.10, one should not trust this result too much, though, because in a simulation with trial-and-error adaption of the demands the resulting steady state price depends also T_p: As the system evolves towards equilibrium "from above", the "recovery time" which is needed for the transient (upward or downward) jumps in utility to die out after a price change gets ever longer and eventually becomes larger than T_p. Hence, no further deflation of the prices is possible because T_p timesteps after a lowering of the price the agents always find themselves at a lower utility level than before.

In the next section, the agents are allowed to *calculate* their optimal demands. This removes the problem with the transient jumps of the utility, since demands adapt almost instantly (delayed only by the redistribution of money). T_p reduces to about 10 and is no longer dependent on N. Of course, this saves also enormously in computation time.

[5]Recall that every testing of a new demand value needs $2T_x$ timesteps: In the first timestep a new value is chosen. In the second timestep the performance of this new value is evaluated and the agents possibly goes back to the old value. No testing of a new demand is considered in the second timestep. The same applies to the testing of new prices: A whole "testing cycle" always needs $2T_p$ timesteps so that the agent has the chance to come back to the former utility level in case she tested a "bad" price.

2.6 Simulations with calculated instead of adapted consumption

The analytical solution implies and the simulation results (in particular Sec. 2.5.1) confirm that, instead of slow adaptation, agents could set their consumption directly via

$$x_{ij} = \frac{M_i}{p_j^2 \sum_{j \neq i} \frac{1}{p_j}} . \tag{2.24}$$

So a possibility is to use this directly in the simulations. In fact, apart from different fluctuations, simulation results look the same as before. This saves enormously in terms of computer time, since we can now do price adaption with $T_p = 10$ instead of $T_p = 10^5$ time steps.

Fig. 2.5 shows the result in the system of 10 agents. As the computer time is greatly reduced, also systems with many more agents can be considered, see Fig. 2.6 for a simulation of an economy with 1000 agents.

From a conceptual point, this means that we allow agents to locally use mathematics in order to find better (i.e. locally optimal) solutions. Since we have demonstrated that the same solution can be reached via adaption, this can be seen purely as a shortcut in computation. Note however that this equivalence of adaptation and calculation is only valid as long as there is a unique maximum. Some discussion in how far such a quantitative difference may nevertheless result in a qualitative difference is made in Sec. 2.10.

2.7 Fluctuations in equilibrium

Eq. 2.20 gives the amount of money all agents will end up with if one agent i charges price p_i while all other agents charge p:

$$M_{oth} = \frac{M_{tot}}{N-1} \frac{p + (N-2)p_i}{2p + (N-2)p_i} \tag{2.25}$$

Using Eq. 2.7 we see that the agents will buy

$$x_i \equiv \frac{M_{oth}}{p_i^2 (1/p_i + (N-2)/p)} \tag{2.26}$$

from agent i. To first order in $(p_i - p)$ this is

$$x_i = \frac{M}{p(N-1)} - 2\frac{M}{p^2 N}(p_i - p), \tag{2.27}$$

where $M \equiv M_{tot}/N$ is the average amount of money per agent. We can now find an expression for the fluctuation in production of agent i that is induced by the testing of price p_i: We assume that the system has relaxed to the homogenous situation where all agents (except agent i) charge the steady state price

$$p = p_{hom} \equiv \frac{2^{2/3}\, M}{(N-1)^{1/3}}[6] , \tag{2.28}$$

[6]Note that M and not M_{oth} is used for the average money of the agents in equilibrium. We assume that for every agent who tests a lower price and who therefore lowers the average money of the agents in equilibrium there will be one who tests a higher price and resets this average money to M. This is correct to leading order in this model, but not in the model of chapter 3, see the remarks below Eq. 2.11

and (in accordance with the adaption algorithm described in Sec. 2.2) we write $(p_i - p_{hom})$ as $(f_{rand} - 1)p_{hom}$, where f_{rand} is a random number not larger than 1.1 and not lower than $1/1.1 = 0.909$. Introducing $q_{hom} \equiv M/p_{hom}$, the production in the homogenous situation, the production $q_i = (N-1)x_i$ of agent i becomes:

$$q_i = q_{hom} - \frac{2^{1/3}(N-1)^{4/3}}{N}(f_{rand} - 1) \tag{2.29}$$

In the large N limit the fluctuation $\delta q(N) \equiv q_a - q_{hom}$ is thus given by

$$\delta q(N) = -(2N)^{1/3}(f_{rand} - 1) \tag{2.30}$$

Therefore, in absolute value, fluctuations will slowly increase with N. *Relative* fluctuations are predicted to be constant in the large N limit:

$$\delta q_{rel} \equiv \frac{\delta q(N)}{q_{hom}} \approx 2(1 - f_{rand}). \tag{2.31}$$

Thus, the production can vary at most by 20% up or $\approx 18\%$ down. In particular, it cannot drop to zero.

The result should be more accurate if one assumes that not only one agent changes her price, but that we have N_l agents that lower the price to p_l and N_h agents that increase the price to p_h at the same time. In our model we have

$$N_h \approx N_l \approx r \times N \tag{2.32}$$

where $r \approx p_{trial}/2$ is about 5% (more correctly, r is given by $0.1/1.1/2 = 0.04545$, see the paragraph below Eq. 2.35). If, as before, p_l and p_h are written in the form $p_l = f_l p_{hom}$ and $p_h = f_h p_{hom}$, the result is (in the large N limit)

$$\delta q_{rel,up} = 2(1 - f_l) + r(f_l + f_h - 2) \tag{2.33}$$

for the relative upward fluctuations and (by symmetry)

$$\delta q_{rel,down} = 2(1 - f_h) + r(f_l + f_h - 2) \tag{2.34}$$

for the relative downward fluctuations. Surprisingly, since in our model $f_l + f_h \approx 2$ on the average, the result is about the same as before, i.e. the fluctuations depend only very weakly on r. The reason is of course that $\sum_{k \neq i} \frac{1}{p_k}$ in Eq. 2.7 depends only weakly on r: It is always approximately given by $(N-1)/p$.

2.8 Comparing results of simulations with different parameters

As seen in Sec. 2.6, the simulation with calculated instead of adapted demands can be considered as a mere shortcut leading to the same equilibrium results and (qualitativly) the same behaviour when the system is away from equilibrium. Therefore, although data from simulations with calculated demands is used below, the conclusions that are drawn should also hold for the model with trial-and-error adaption of demands. A discussion of a (here not relevant) *difference* between the two kinds of adaption is made in Sec. 2.10.

First of all, we are interested in comparing simulations with different numbers of agents.
Eq. 2.31 states that in the large N limit the relative fluctuations in production are independent of the number of agents and not larger than $\pm 20\%$ of the equilibrium production value. Fig. 2.7 shows the result of an experimental test of this.
Fig. 2.8 shows how the equilibrium price is decreasing with the number of agents in the system. A comparision with the analytically predicted result $p = \frac{2^{2/3}M}{(N-1)^{1/3}}$ is made.
Fig. 2.9 is an illustration of the fact that the average amount which the agents sell is decreasing with increasing N, the average production is increasing with N, compare Sec. 2.4
Fig. 2.10 shows that utility is increasing with N. In this model, for large N, utility is proportional to $N^{2/3}$ (see Eq. 2.14). This unbounded increase of utility with N is understandable for a model with square root consumption utility as the slope of the consumption utility is infinite for $x \to 0$. The situation is very different in the logistic model of chapter 3 where substitution is always at hand: strong downward fluctuations result from the testing of prices and average utility even *de*crease with N. Even if price-testers are excluded from the calculation of the average utility, utility is bounded for $N \to \infty$: In this limit, the slope of the logistic consumption utility is 1, which means that the agents do not gain more utility if they can buy from a larger number of agents.

The following simulations explore how the model reacts to different amounts of money in the system. The number of agents is kept constant at 10. Fig. 2.11 shows that production and utility are *independent* of the amount of money in the system, while prices are proportional to it.

Finally, Fig. 2.12 is an exponential fit to the decay curve of the average price in the system of 1000 agents. The simulation starts at a much too high average price $P_{av,0}$ of about 20. The average price decays exponentially according to $P_{av,t} = P_{av,0}\exp(-ct)$ until P_{eq} is reached. Exponential decay is expected since the mutation is multiplicative. The smooth crossover to the equilibrium value (instead of a sharp corner) is caused by the fluctuations. The fitting parameters are:

$$P_{av,0} = 19.99, c = 0.000237, P_{eq} = 0.159. \tag{2.35}$$

These values are consistent with the price adaption parameters: On the average, about 91[7] agents test a new price every T_p timesteps. Half of them will choose to test a lower price according to $p \to p/f_{p,rand}$. The expectation value $\langle f \rangle$ of the reduction factor $1/f_{p,rand}$ is

$$\langle f \rangle = \frac{\int_1^{1.1} \frac{1}{x} dx}{0.1} = 0.9531. \tag{2.36}$$

This results in $c \equiv ln(\langle f \rangle)/(T_p/(0.091/2)) = 0.000218$.

2.9 Separation of time scale and stability

The stability of the above model hinges critically on the fact that adaptation of prices is much slower than adaptation of demands. In effect, price adaptation is made so slow that demands (and money) adaptation has always completed before the performance of a price change is evaluated. As shown in Sec. 2.5.1 by direct simulation we need to wait about 50000 timesteps until demands and money have adapted to a new price situation. This figure is correct for the considered systems of 10 and 50 agents, but generally it scales with N (see also the comments about it in the next section). If demands are

[7] Every T_p timesteps $p_{trial} = 10\%$ of the agents that did *not* test a new price in the last testing period enter trial mode. Asymptotically this means that there will be $(0.1/1.1)N = 0.0909N$ agents that test a new price every T_p timesteps.

calculated as in Sec. 2.6 the adaption time is greatly reduced to about 5 timesteps and it is therefore enough to wait only $T_p = 10$ timesteps between two consecutive price changes. It is very important to note that changes in price must still be made on a slower time scale than adaption of demands as still some time is needed for the redistribution of *money*. Note also that in the second case T_p is independent of N , while in the first case T_p scales linearly with N.

In terms of stability, it is *not* possible to violate this rule: If demands are calculated, prices will go to infinity if $T_p \leq T_x$, because in the timestep after a price increase there is an upward jump in utility. In the case of trial-and-error adaption of demands, making price adaptation too fast relative to consumption adaptation leads to higher steady state prices or (in case of very fast price adaption) to continuously increasing prices[8]. The reason for this is simple and will be explained in more detail in the next section: A price increase will initially increase the corresponding income; the reduction in sales sets in later.
The tabular below gives a quantitative overview on the dependence of p_{SS} on T_p for too low choices of T_p in a system of 50 agents with trial-and-error adaption of demands:

T_p	p_{SS}
100000	0.46
80000	0.48
60000	0.51
20000	1.31
10000	6.5
1000	$2e8$
1	∞

(2.37)

The separation of time scales is also reflected in the Lagrangian solution: One has to solve for the x_{ij} first, when prices are given, and then optimize prices based on the consumption behavior.

2.10 A difference between trial-and-error and calculated adaption of demands

Fig. 2.13 is a comparision between the simulations with trial-and-error adaption and the corresponding simulations with calculated adaption of demands. Obviously, the average equilibrium price p_{SS} is higher for trial-and-error than for calculation. Why are the results (apart from different fluctuations and different timescales) not the same? The reason is that in the case of trial-and-error adaption the system exhibits a feature which is absent when the demands are calculated:
In the case of trial-and-error adaption every test of a new price is followed by a transient jump of utility. If a lower price is tested, the jump will be downwards , because in the first timesteps after the price change the agent earns less money for the same amount of work. Mathematically, assume that all agents charge price p and one agent changes it to $p + dp$. The production $q \approx M/p$ of the agent will not change in the first timestep, but her income has changed to $q(p + dp)$. Thus she can buy now

[8]This is clear for example for $T_p = T_x$ (for the same reason as for calculated adaption). The transition point from “too high” to “infinite” equilibrium price can be described like that: If T_p is so small that the recovery time after an average price increase is longer than T_p even if there would be no noise, all average (and larger than average) price increases will be accepted an thus prices will go to infinity.

$x_{new} \approx \frac{q(p+dp)}{N-1}$ from the other agents. The resulting jump in utility is

$$dU \approx 2(N-1)(\sqrt{x_{new}} - \sqrt{x_{old}}) \quad (2.38)$$

where $x_{old} \approx M/p/(N-1)$. To first order in dp this is

$$dU = \sqrt{M(N-1)}\frac{dp}{p}. \quad (2.39)$$

Consider the case $dp < 0$ now. Due to adaption of the demands the income will slowly go up (along with the production) and (if one waits long enough) a new steady level of utility will be reached. T_p must be chosen large enough so that this level is reached before T_p timesteps are over. Fig. 2.14 shows how these transient jumps look like in case all agents charge p except one with $p_l = p + dp < p$. Different values for p and p_l are considered in a system of 50 agents.
Obviously, the "recovery time" (the time it takes for the agent to come back to the former level of utility) shows two important features. First of all, for a given p, the recovery time does to a good approximation not depend on dp (of course we consider only p_l's that are about $1/1.05p$). A higher $|dp|$ causes a stronger jump in utility, but it gives also a stronger signal that the other agents should increase their demand for that product.
Secondly, a dependence of the recovery time on p is observed. As $p \to p_{SS} \approx 2^{2/3}M/(N-1)^{1/3} = 0.434$, the recovery time increases and eventually becomes larger than $T_p = 100000$. This effect is easily understood. As $p \to p_{SS}$, the average production increases. But because of the concavity of the consumption utility, the agents gain ever less from a further increase of the demands. Thus, the signal that the agents should increase their demand for the cheaper product becomes weaker and more hidden by the (constant) noise (the constant testing of all demands) in the system. Indeed, in a simulation where the noise is reduced, the recovery time stays very small as $p \to p_{SS}$, see Fig 2.15.
Given these two features of the recovery time, one sees that as soon as the recovery time becomes larger than T_p, the deflation of prices stops. At this point, an average price increase has about the same chance to be accepted as an average decrease. As explained in the previous section, if T_p is very small, even a constant inflation of the prices can result. In this sense, T_p can be regarded as a kind of "patience-parameter" for the agents. It gives the maximum time the agents are willing to wait after a downward jump of utility to come back to the former "standard of living".

In the simulations with calculated demands there are also transient jumps in utility, but the recovery time is not dependent on either N or the current price level in the system. The transient has died out as soon as the redistribution of money has taken place (see the previous section). It is clear then, that the resulting steady state price can only be lower or equal to the steady state price in the trial-and-error simulation.

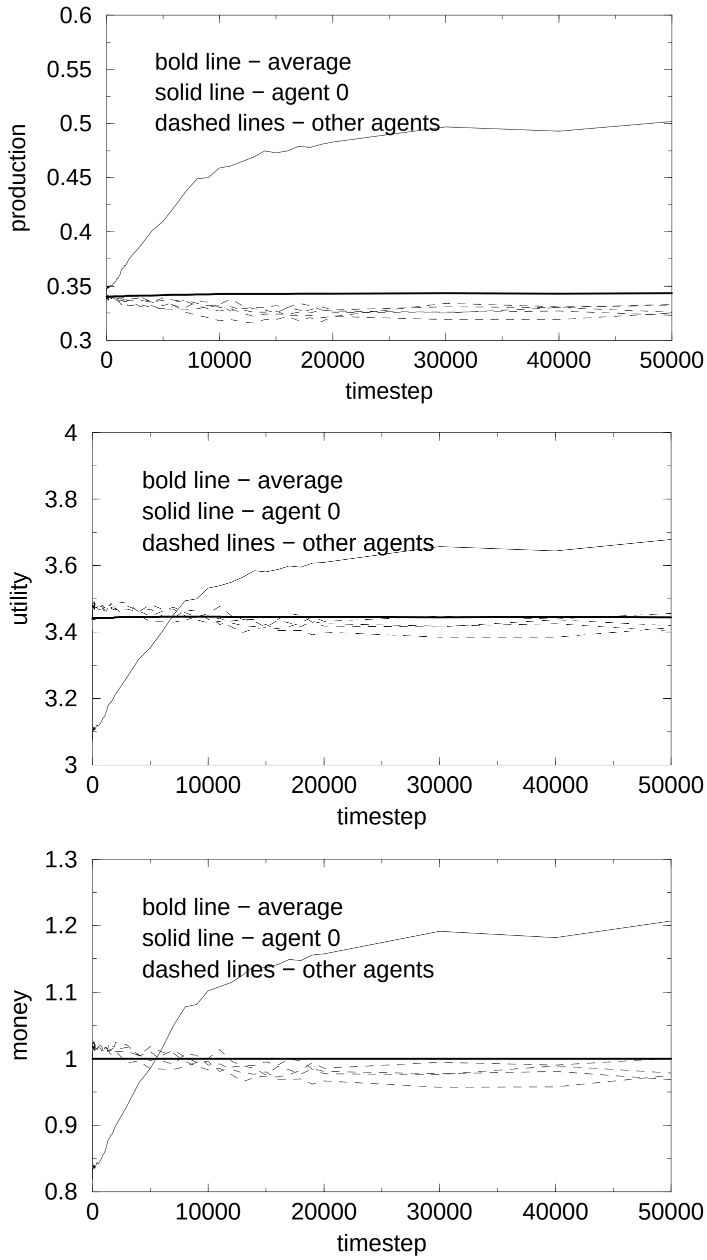

Figure 2.1: *Simulation where the demands $\hat{x}_{ij}$ adapt via trial-and-error while prices are fixed. The price of agent 0 is fixed at 2.4, the prices of all other agents are fixed at 3. TOP: Production. CENTER: Utility. BOTTOM: Money. The fat line describes the average behavior, the single thin line the behavior of the one agent with the lower price, all other thin lines are examples of other agents. One clearly sees that demands slowly move towards the good with the lower price. One also sees that this results in a higher utility for that agent, meaning that in this case average prices are too high and an individual has an advantage when charging a lower price. This effect is what will drive the adaptation of prices in Sec. 2.5.2.* $N = 10$, $M = 1$.

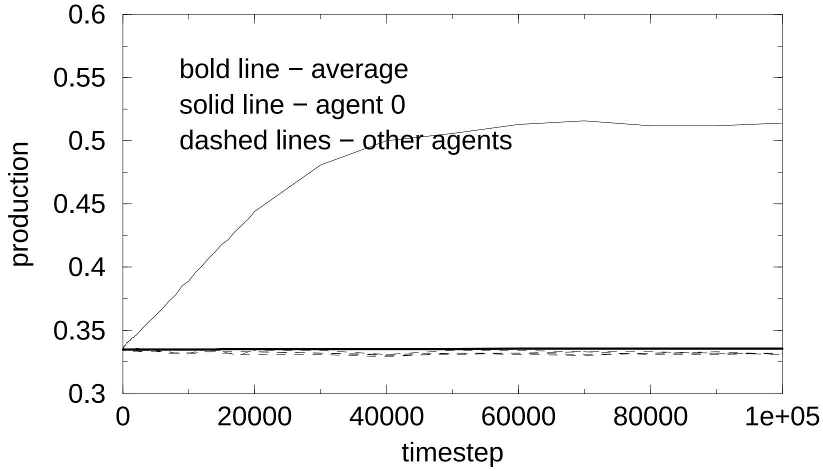

Figure 2.2: *Same simulation as in the previous figure, but with 50 instead of 10 agents. 100000 timesteps for the adaption of the production to the price-situation is still enough.*

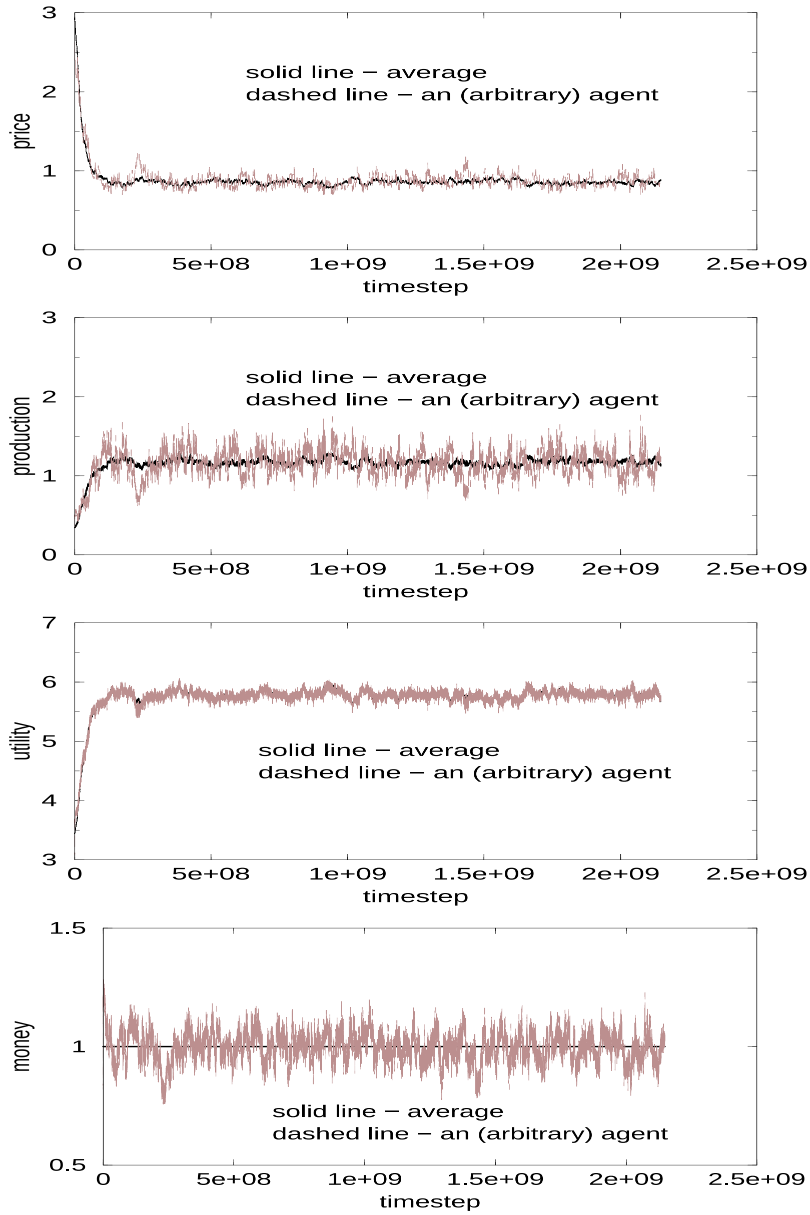

Figure 2.3: *Simulation where both the demands* $\hat{x}_{ij}$ *and prices* p_i *adapt via trial-and-error, system of 10 agents. TOP: Prices. SECOND: Production. THIRD: Utility. BOTTOM: Money. One sees how the system relaxes towards a price of about* 0.87*, a production of about* 1.15*, and a utility of about* 5.7*. The bottom figure shows that the average per capita amount of money in the system remains at one (as it should) but that the individual amount of money fluctuates considerably.*

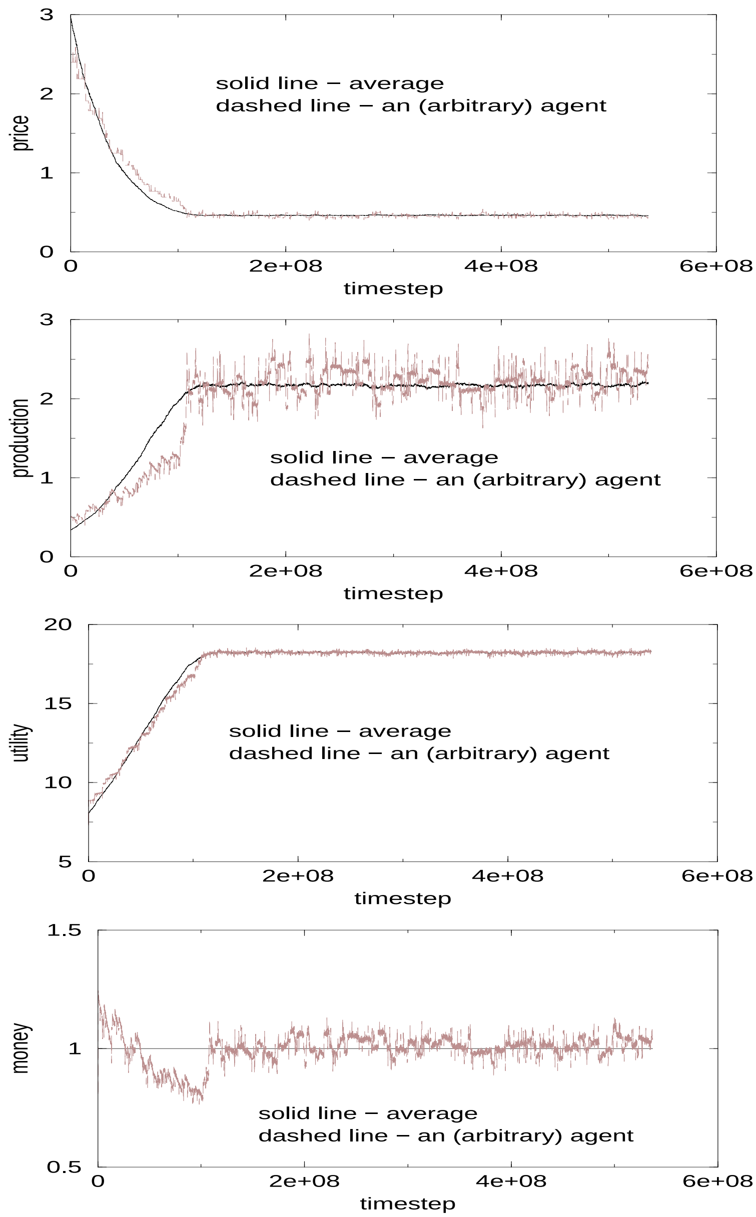

Figure 2.4: *Same simulation as in Fig. 2.3, but with 50 instead of 10 agents. The system relaxes towards a price of about* 0.46*, a production of about* 2.18 *and a utility of about* 18.22*. This can be compared to the analytical result (see Sec. 2.4) of* 0.43*,* 2.31 *and* 18.59*, respectively. Note that the relative fluctuations in the individual production are about the same as in the simulation with 10 agents.*

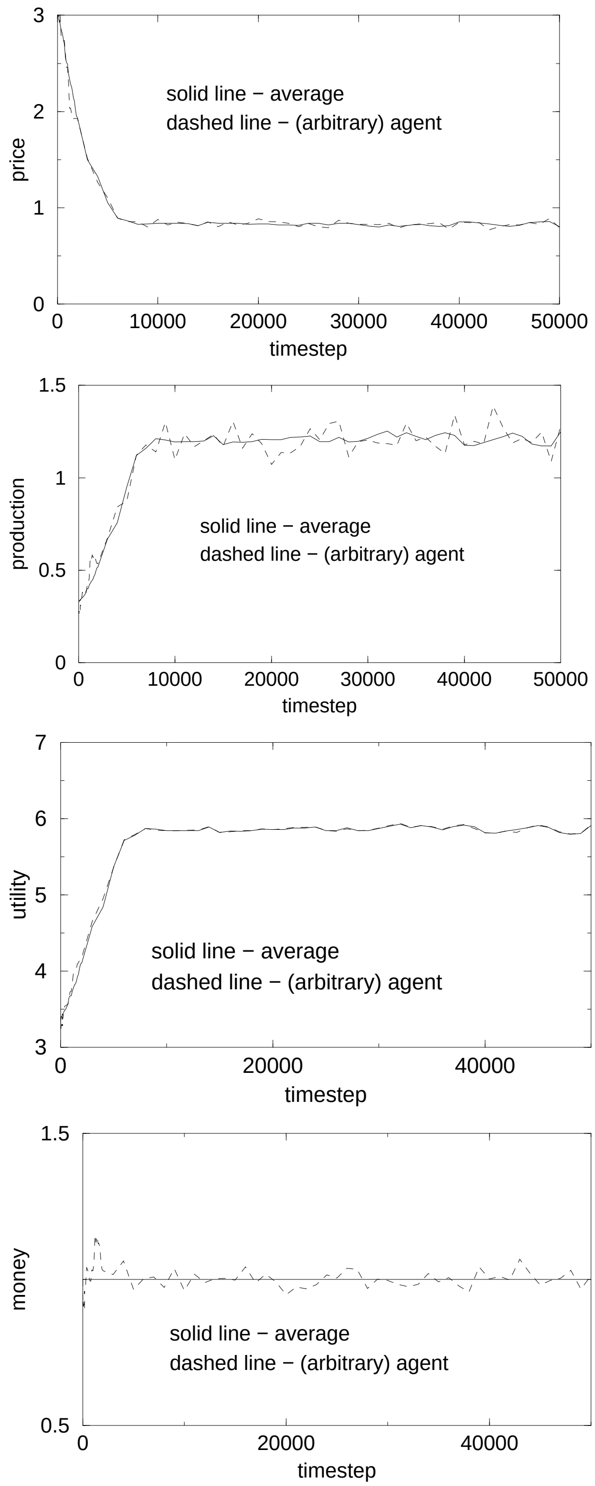

Figure 2.5: *Simulation where demands are calculated optimally while prices adapt via trial-and-error. TOP: Prices. SECOND: Production. THIRD: Utility. BOTTOM: Money. One sees that the system relaxes towards the same values as in Fig. 2.3. Note the much shorter time scale on which that happens.*

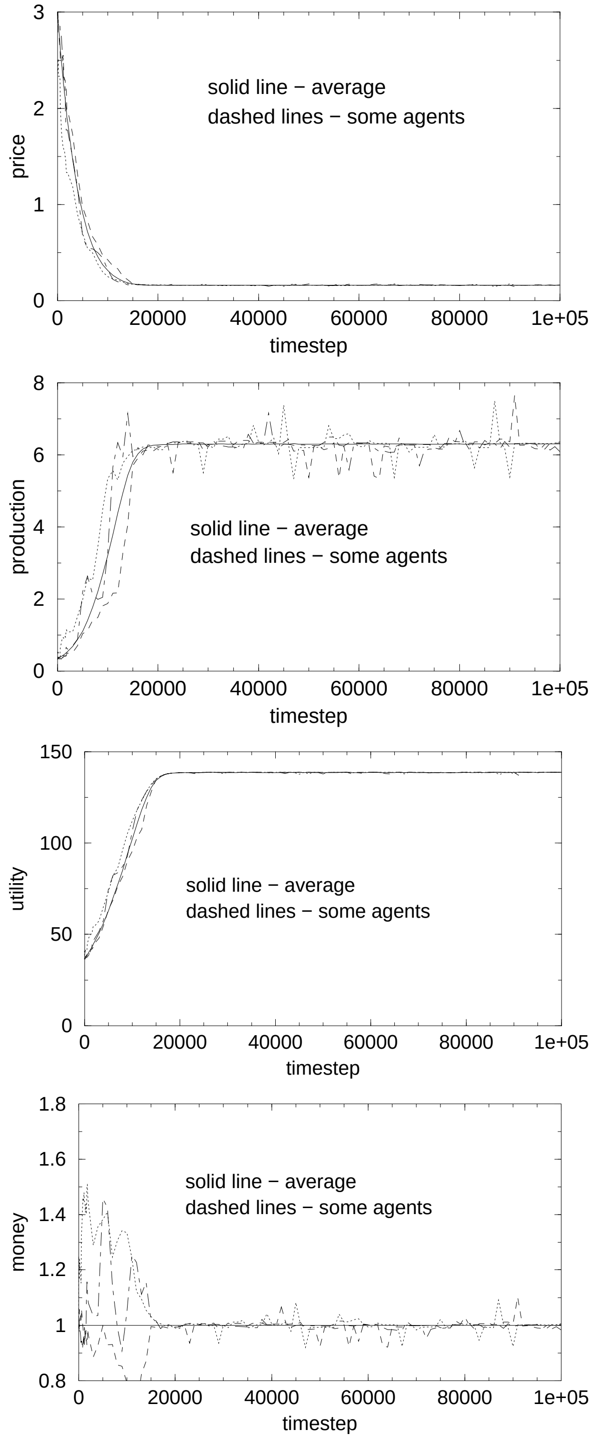

Figure 2.6: *Simulation where demands are calculated optimally while prices adapt via trial-and-error, with 1000 agents. TOP: Prices. SECOND: Production. THIRD: Utility. BOTTOM: Money.*

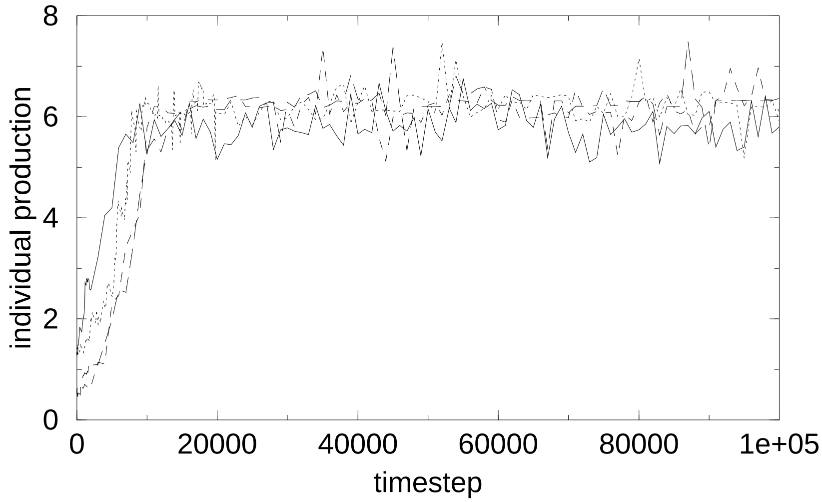

Figure 2.7: *In the large N limit the relative fluctuations are independent of the number of agents. The figure shows the individual production of an arbitrary agent in systems of 10 (solid line), 50 (dotted line), 100 (dashed line) and 1000 (long dashed line) agents. The results are all scaled up to the* $N = 1000$ *production level. The fluctuations should not be larger than* 20% *of the equilibrium production value, which is confirmed by the figure, too. Note especially that as a consequence, in this model the production of an agent can never drop to zero.*

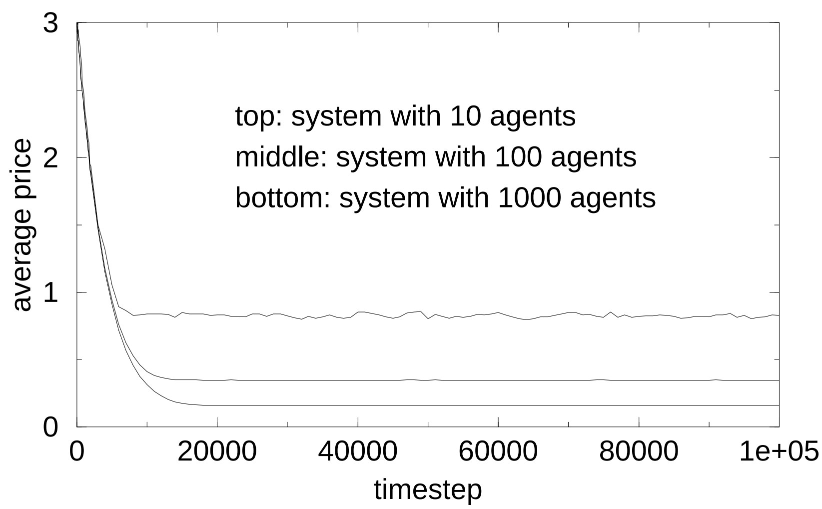

Figure 2.8: *The equilibrium price is a decreasing function of the number of agents in the system. The decrease in price is in agreement with the "large* N*" solution from Sec. 2.4.2:*

N	10	100	1000
simulation	0.824	0.345	0.159
analytical	0.763	0.343	0.159

(2.40)

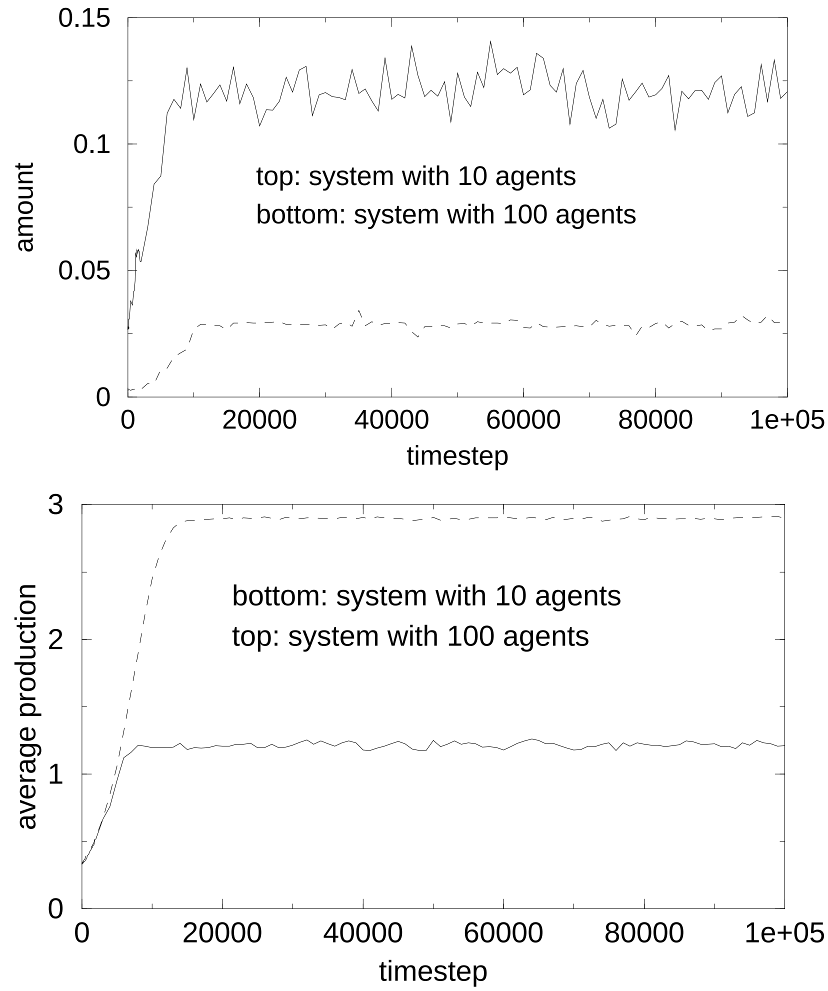

Figure 2.9: *The average amount which the agents sell is decreasing with N (as it should), the average production is increasing with N.*

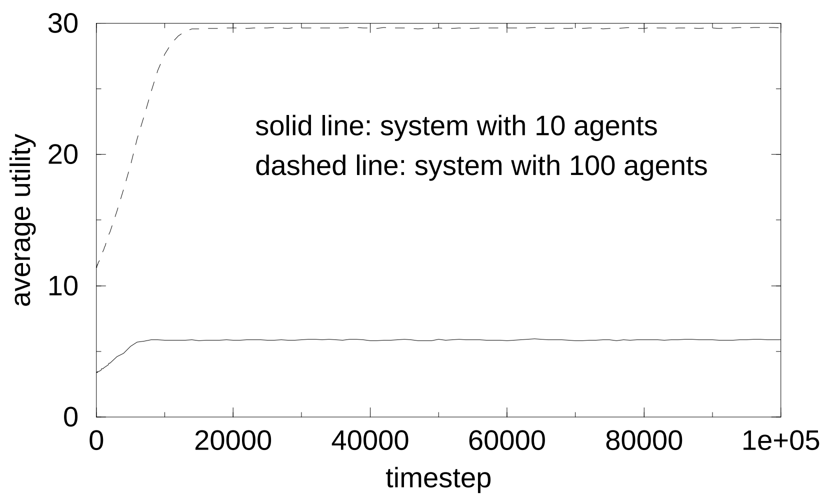

Figure 2.10: *Average utility is increasing with N. Compare with the equivalent Fig. 3.12 for the logistic model where a very different behaviour is observed.*

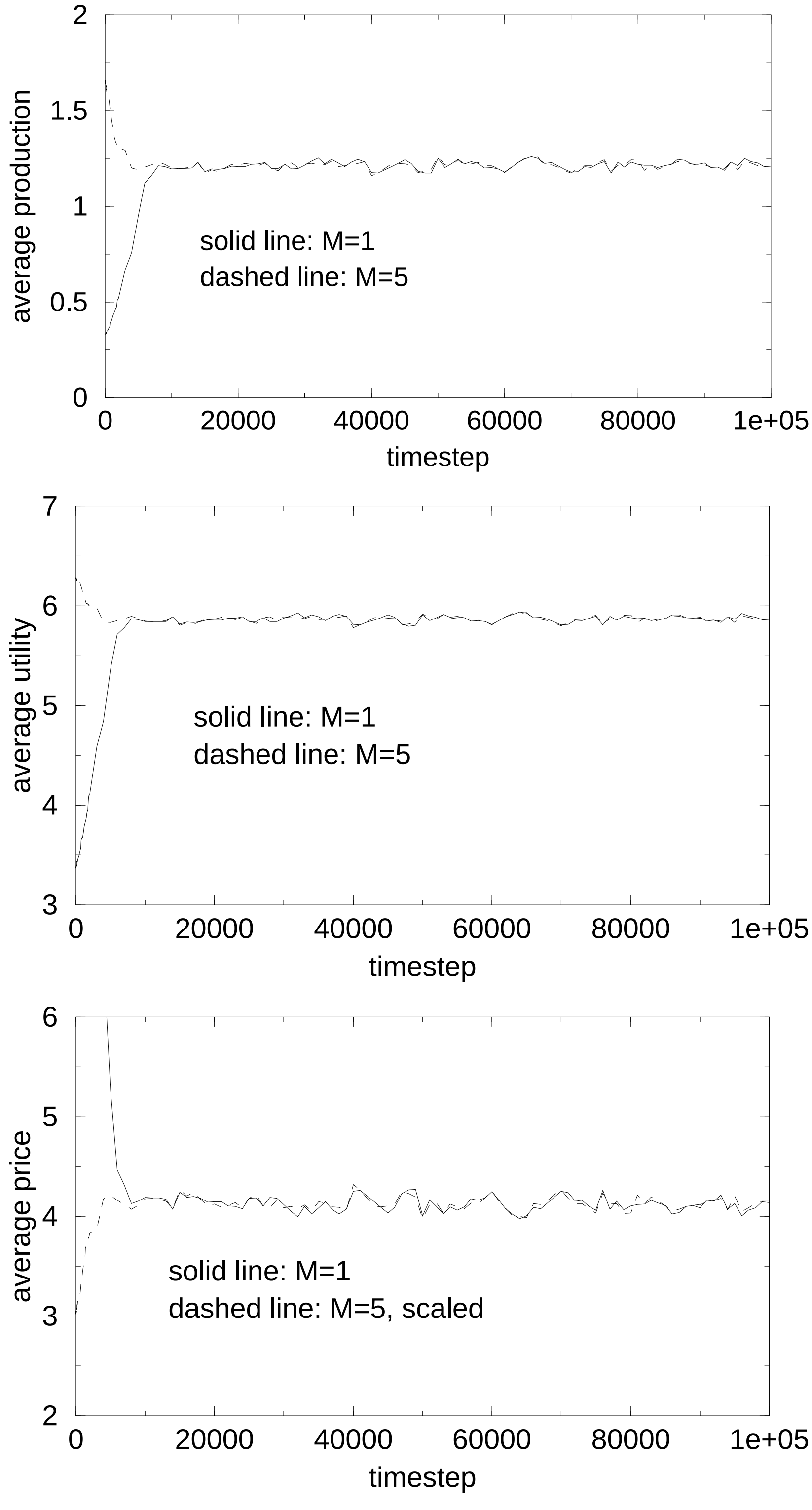

Figure 2.11: *The production and utility are independent of the amount of money in the system, but prices are proportional: For the prices, the solid line (which represents the simulation where the average amount of money is* 5*) is scaled by a factor of 5. For production and utility no scaling was applied.*

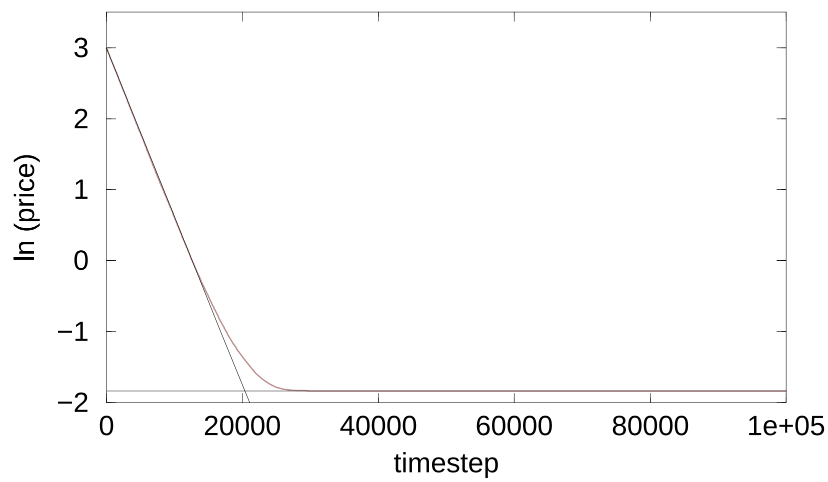

Figure 2.12: *Exponential price decay until equilibrium price is reached. Fitting parameters:* $P_{av,0} = 19.99, c = 0.000237, P_{eq} = 0.159$*. Simulation with 1000 agents and a high initial price level of about 20.*

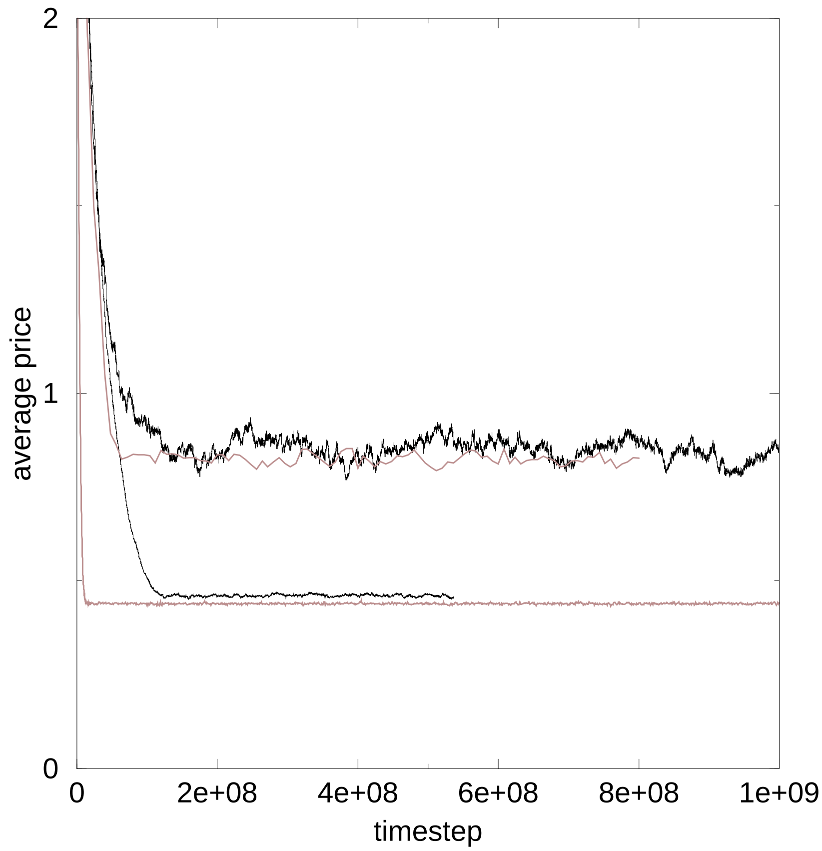

Figure 2.13: *Comparision of the result for the steady state price level in the simulations with trial-and-error adaption (solid lines) and with calculated adaption (thin lines) of demands. Top curves are for $N = 10$, bottom curves for $N = 50$. The curves for trial-and-error give a higher result than the corresponding curves for calculated adaption. The t-values of the thin curves are scaled, as these simulations run for a much shorter time.*

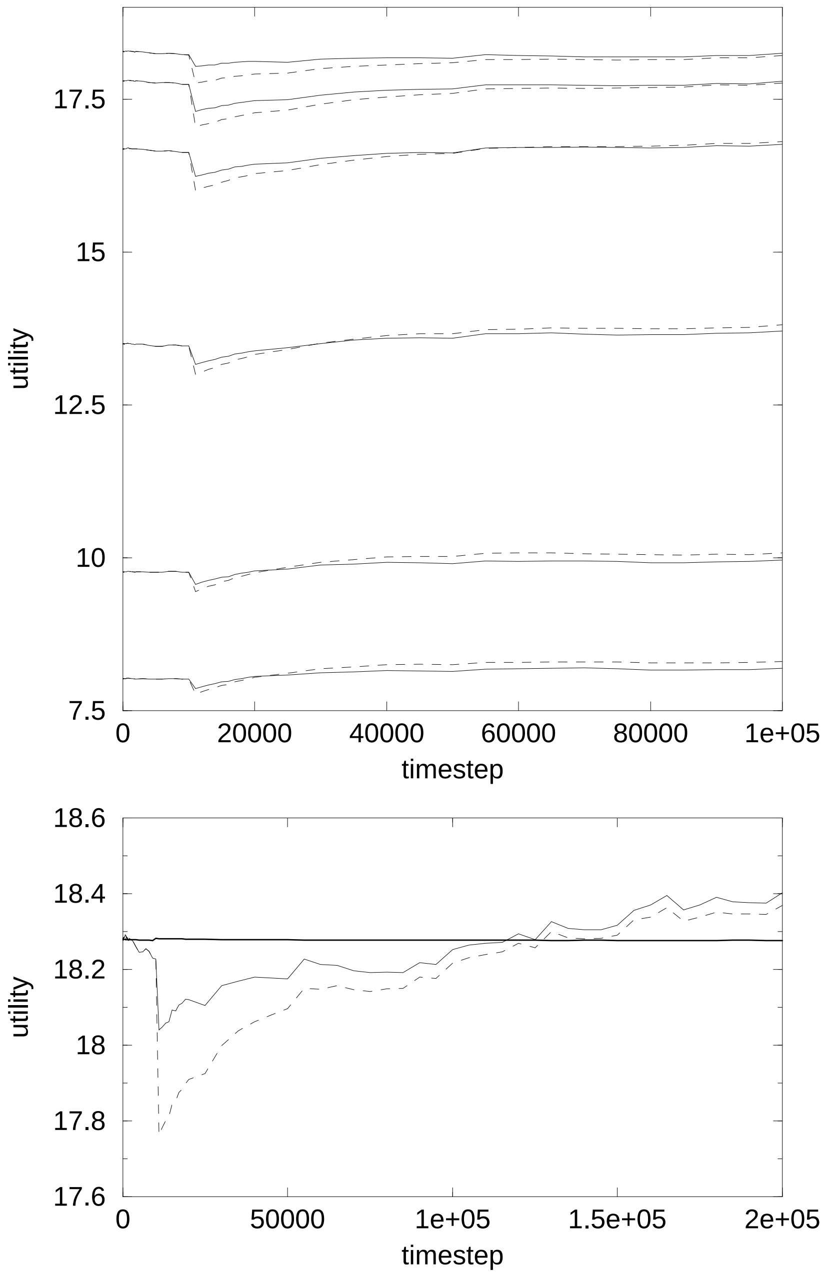

Figure 2.14: *Dependencies of the "recovery time". Simulation of systems with* 50 *agents and trial-and-error adaption of the demands. TOP: The prices are fixed at (from bottom to top)* $p = 3, 2, 1, 0.6, 0.5, 0.46$. *At timestep* 10000 *the agent whose utility is shown lowers her price to* $p_l = p/1.05$ *(solid lines) or* $p_l = p/1.08$ *(dashed lines). (For the simulation with* $p = 0.46$ *it is* $p_l = p/1.02$ *and* $p_l = p/1.05$*, respectively.) The recovery time increases as* p *decreases, but it is approximately independent of the change in price for a given* p*. For* $p = 0.5$ *the recovery time is approximately equal to* 90000 *so that acceptance of lower prices is strongly suppressed. For* $p = 0.46$*, the recovery time is larger than* $T_p = 100000$ *and no further deflation of the prices will be accepted anymore. BOTTOM: The simulation with* $p = 0.46$*. The fat solid line is the average utility level, the recovery time is the time between the jump and the crossing of individual and average utility.*

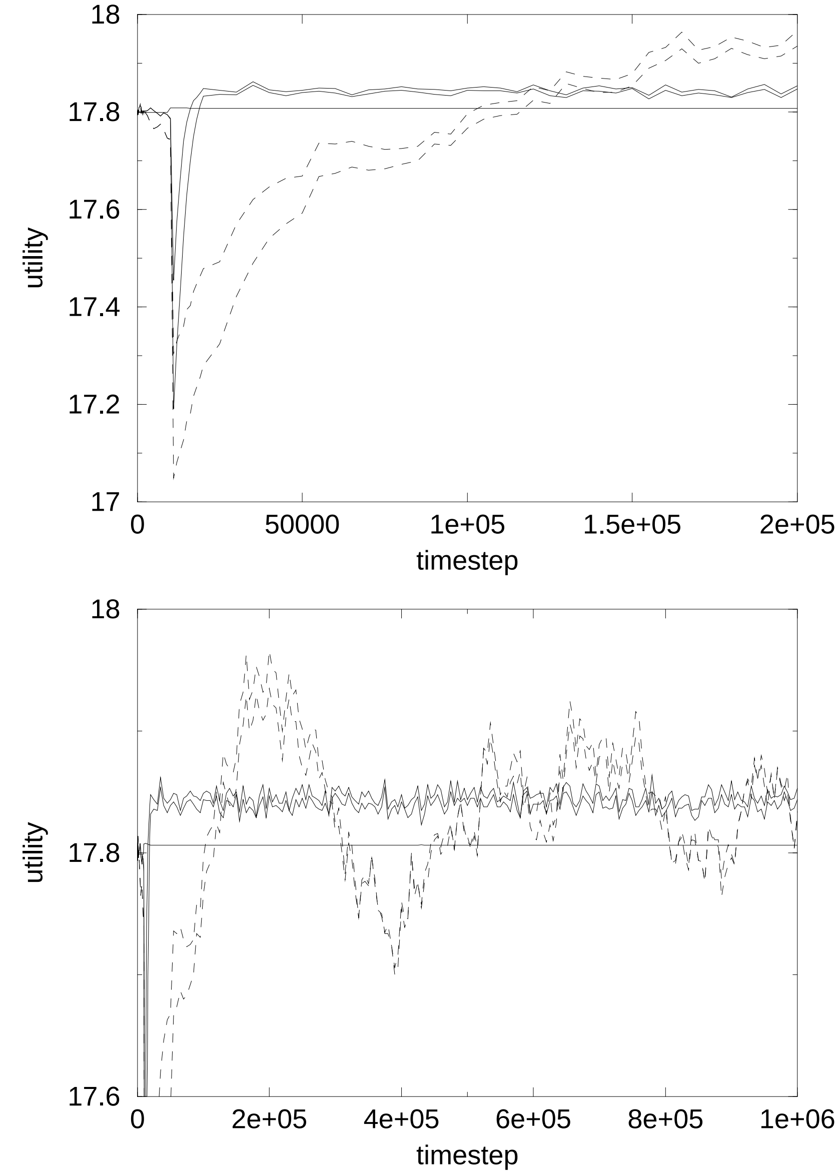

Figure 2.15: *The influence of the noise (the constant testing of demands) on the recovery time. The dashed lines are the same simulations as in Fig. 2.14 with* $p = 0.5$ *and* $p_l = p/1.05$ *and* $p_l = p/1.08$*, respectively. In the simulations represented by the solid lines, the same parameters are used, but all agents are only allowed to adapt the demand for the cheaper product. They always have to buy the same amount from all the other agents. The constant solid line is the average utility. TOP: The recovery time is much shorter when the adaption of demand of the cheaper product is not desturbed by the noise in the system. The recovery is then governed by the parameters* T_x *and* $f_{x,rand}$ *(see Sec. 2.2) alone, i.e. during recovery on the average every agent will raise the demand by a factor* $\langle f_{x,rand} \rangle = 1.005$ *every* $2 \times (2T_x) \times N = 4N$ *timesteps. Thus, there is no dependence on* p *like it is described in Fig. 2.14. BOTTOM: Also after the recovery time has passed, the noise makes it very difficult for the agents to find the optimal demand for the product.*

Chapter 3

Logistic consumption utility

3.1 The model

In the following we will consider the effects of a different functional form for the consumption utility, i.e.

$$U_i = -\frac{q^2}{2} + \sum x_{ij}\left(1 - x_{ij}\right). \tag{3.1}$$

The difference to the square root utility function is that in the limit $N \to \infty$ (where $x \to 0$), the slope of the consumption utility is 1 and not ∞. This means that it is very easy here to substitute one product for another, e.g. if one considers the price of a product too high. It is very clear therefore that all agents which do not behave like the average will be in serious trouble, as they will either lose all their money or they will have to work much too much. Thus, testing of prices causes very large fluctuations which makes the model harder to handle than the last: If one is not very careful it will always be possible that the economy completely "loses" some of its agents. Furthermore, it is not always possible to use the same measurement methods (e.g. for equilibrium utility) as in chapter 2, because the fluctuations can be so large that they are no longer symmetric with respect to upward and downward fluctuations.

3.2 Analytic results

3.2.1 Optimal consumption

As in the first model, the optimal consumption given $M_{i,t-1}$ can be calculated analytically. The result is

$$x_{ij} = \frac{1}{2}(1 - \lambda_i p_j) \qquad \text{and} \qquad \lambda_i = \frac{\sum_{k\neq i} p_k - 2M_{i,t-1}}{\sum_{k\neq i} p_k^2} \tag{3.2}$$

Altogether:

$$x_{ij} = \frac{1}{2} - p_j \frac{\sum\limits_{k\neq i} p_k}{2\sum\limits_{k\neq i} p_k^2} + M_{i,t-1}\, p_j \frac{1}{\sum\limits_{k\neq i} p_k^2}. \tag{3.3}$$

Note that x_{ij} is not proportional to $M_{i,t-1}$. In the homogeneous situation, all prices are the same and thus the first and the second term cancel out, but this is no longer true when prices are different. In this case, if an agent happens to have the double amount of money to spend in a certain timestep than in the previous, she will not buy the double amount of goods from every agent, but distribute her money in a much more complicated way.

According to the above equation, x_{ij} can become negative. For example, assume $M_i \to 0$ and all prices except p_m the same. Then

$$x_{im} \approx \frac{1}{2} \left(1 - \frac{p_m^2 + (N-2)\, p\, p_m}{p_m^2 + (N-2)\, p^2} \right) , \tag{3.4}$$

which is negative for $p_m > p$.

As usual, in such cases we will set the corresponding x_{im} to zero. This means, however, that the corresponding good needs to be excluded from the calculation; in other words, we now need to calculate $\sum_{k \neq i,m} p_k$ and $\sum_{k \neq i,m} p_k^2$ and re-calculate *all* $x_{ij, j \neq m}$. This needs to be done iteratively until no negative x_{ij} is left. Note that this is not an issue in equilibrium when all numbers are "reasonable", but it is an issue for a simulation.

3.2.2 Optimal price

The "large N" result can be obtained exactly as in basic model. It is here

$$p \approx \frac{(N+3)M}{N-1} \tag{3.5}$$

which means that

$$x \approx \frac{1}{N+3} \text{ and } q \approx \frac{N-1}{N+3}. \tag{3.6}$$

This gives for the utility:

$$U = \Big(\frac{1}{N+3} \Big)^2 (1/2 N^2 + 2N - 2.5), \tag{3.7}$$

which is a monotonically increasing function of N with a limiting value of 0.5 (see the plot of this function, Fig A.2). Note the difference between this result and the result $U \propto N^{2/3}$ for square root consumption utility. Here, because substitution gets ever easier for $x \to 0$, the utility is bound in the limit $N \to \infty$.

As will be explained in Sec. 3.4 one has to be careful about the interpretation of M in Eq. 3.5. Especially for large N, M cannot be set equal to M_{tot}/N.

3.3 Simulation results

3.3.1 Trial-and error adaption of the demands

Figs. 3.1 and 3.2 show the results when both demands and prices adapt via trial-and-error, as described in Secs. 2.1 and 2.2 for the square root consumption utility.

Because of the strong reactions which are provoked by the price adaption procedure, two things had to be changed compared to Secs. 2.1 and 2.2:

The first change has to be introduced in order not to "lose" some agents: During the time when the system is evolving towards the steady state and the average price is constantly falling it can happen that an agent does not try to adjust her price downwards for a long time and therefore ends up with a price which is much higher than the average. Of course, this can happen in the first model, too, but only with logistic consumption behaviour these agents will hence get a completely wrong signal to where they should adapt the price!

The crucial difference is that here, the other agents would optimally buy nothing at all from these agents and rather spend all their money for other (less expensive) products: The optimal consumption of this product as given by Eq. 3.3 would be negative. So if the price p_h of the "expensive product" is high enough, the other agents will constantly decrease their demand x_h for it, no matter in which direction this agent changes her price. Note that x_h cannot drop to zero, though, as the adaption is multiplicative. Thus, the utility of that agent slowly decays to zero (exponentially) and it seems that the agent would always go back to the same price after testing.

The stochasticity (noise) in the model leads to a different behaviour: Suppose the price for the product stays constant. Then, as x_h goes to zero, the effect of a further decrease of x_h on utility is hidden ever more by the (constant) noise in the system, as, in absolute value, the decreases must become smaller and smaller. The agents will thus slowly lose the signal that they should decrease their demand x_h of the product. Now suppose the agent changes her price by dp. This will cause a jump in utility in the next timestep by

$$dU = q * dp + O(q^2) \tag{3.8}$$

(The calculation to obtain this result is the same as in Sec. 2.10, but the expansion is for small q, as $q \to 0$. dp need not be small). After that, no matter if dp is larger or smaller than 0, utility will still have the tendency to go down. As this tendency becomes ever weaker, the effect of a price change will be (on a timescale of the order T_p) a mere shift of utility up or down by dU. Thus, price *increases* will be accepted, because the signal (the tendency of the demand to go down) is so weak. The result will be that p_h steadily goes up, the increase being damped only by the condition that the signal to decrease the demand must not become so strong that the transient jump in utility after a price increase can die out before T_p timesteps are over.

The following four figures should serve as an illustration of this effect:

Fig. 3.3 shows the result of a simulation where nothing is done to avoid this "loss of agents". The price history of one agent who meets the "fate" described above is shown. Note that the system relaxes towards a higher average price than in Fig 3.2. The reason is that because the system has lost some of its agents, the rest can share a larger amount of money per agent, which causes the steady state price to go up, see Eq. 3.5.

In the next simulation (Fig. 3.4), all agents exept one have a fixed price $p = 2$. One agent starts with $p_h = 5$ and is allowed to adapt her price. The figure shows that this agent receives a completely wrong signal: The agent chooses an exponential increase of her price instead of going to (and even a bit below) the price level of the other agents. The reason why the price increases are accepted is shown in Fig. 3.5: As the demand of the product goes to zero, the signal to further decrease it becomes very weak. If the prices are kept fixed at $p = 2$ and $p_h = 5$, and the effect of a single increase of p_h is studied for different initial values $x_{h,0}$ of the demand, one sees that if $x_{h,0}$ is small enough, the price increase does merely shift up the utility by dU (on a time scale of order T_p).

Fig. 3.6 shows that if we reduce T_p, the inflation of the price p_h (per T_p) will be faster. Of course, this is because price increases have better chances to be accepted. This is the same effect as the one which causes the trial-and-error simulations to yield a higher equilibrium price than calc simulations: smaller T_p means less time for the transient jump of utility to die out.

In order to prevent this unreasonable behaviour it was enough to introduce a "lower production limit": Every agent that produces less than 10% of the average production is forced to lower her price as long as her production stays below this value.

The problem that high price agents get decoupled from the system will be present in the next section where agents are allowed to calculate their optimal demands, too.

In the system of 50 agents a second change has to be made, because $T_p = 100000$ is not enough to get a reasonable result for the equilibrium price. As explained in Sec. 2.10, T_p must be chosen so large that even as $p \to p_{SS}$ (as an approximate measure of p_{SS} Eq. 3.5 can be used) the transient jumps in utility after a change in price have the time to die out. There are essentially two parameters that determine the length of the recovery time after a utility-jump:

First, because of the concavity of the consumption utility, the influence of the constant noise in the system causes the recovery time to increase as the average price in the system decreases. This effect determines for example how fast the utility goes up after a downward jump of utility due to a lowering of the price. The strength of this effect should be about the same for any utility function: E.g. for logistic consumption the system reacts stronger to price changes but this does not only increase the "background noise" that dampens the recovery but it means also that there will be a stronger signal for all agents to react to the price change under consideration which makes the recovery faster.

The second parameter is $dU(dp)$, i.e. how much the utility jumps after a change dp of the price. Performing the same calculation as in Sec. 2.10, one sees that the jump in utility is here

$$dU = \frac{M(N-1-2M)}{N-1}\frac{dp}{p}. \tag{3.9}$$

For a comparision with the result Eq. 2.39 we have to consider the relative jump lengths $dU/U(p)$. If the price level in the systems are p_{sqrt} and p_{log}, respectively, the demands are approximately given by $x = M/p/(N-1)$ and the production by $q = M/p$ (subscripts omitted). This gives the utility levels $U(p)$ after inserting q and x into the respective functional forms of the utility. Now assuming dp/p to be the same in both cases we can look at the quotient of the prefactors, i.e. we consider

$$Q \equiv \frac{dU_{log}(p_{log})/U_{log}(p_{log})}{dU_{sqrt}(p_{sqrt})/U_{sqrt}(p_{sqrt})}. \tag{3.10}$$

We choose p_{sqrt} and p_{log} to be of the form $c \times p_{SS,analyt}$, where c is between 1 and 2 and $p_{SS,analyt}$ is the respective analytical result for the Nash equilibrium price. Like this, the above quotient becomes a function of c and N. The functional form of it is quite complicated, but the result is

$$Q \approx 4 \tag{3.11}$$

for the above range of c and N between 10 and 100. Thus, the jump in utility will be about 4 times larger for logistic than for square root consumption utility. For the simulation with 50 agents, $T_p = 100000$ is no longer enough then, as is shown in Fig. 3.7: In this simulation, the prices are fixed at about 40% above the respective analytically predicted equilibrium values (i.e. $c = 1.4$) and at time $t = 10000$ one agent changes her price according to $p \to p_l = p/1.02$. Clearly, the recovery time for logistic consumption behavior is much longer than for square root consumption behavior because the jump in utility about Q times larger. It is also clear from the figure that T_p must be chosen much larger than 100000 for logistic consumption, as even for this relatively high price level, the lower price would not be accepted after 100000 timesteps (as explained in Sec. 2.10, the recovery time will become even longer as p decreases further). For the actual simulation (Fig. 3.2) we chose $T_p = 500000$. Fig. 3.8 shows the result of a simulation with $T_p = 100000$ compared to the result of the simulation of Fig. 3.2. As T_p is much too small, the deflation of the prices stops too early which means that the system ends up with a higher equilibrium price than in the simulation with $T_p = 500000$ or in the simulation where demands are calculated.

3.3.2 Calculated adaption of the demands

Figs. 3.9 ($N = 10$) and 3.10 ($N = 100$) show the results when demands are calculated according to Eq. 3.3 instead of adapted by trial-and-error. This includes the iterative procedure to replace negative x_{ij} by zero. Again, we note that calculating instead of adapting consumption gives (within the limits set by too small T_p in the trial-and-error simulations) the same result while speeding up the simulation.

An important observation is that in the simulation the prices of some of the agents perform a random walk. The problem is related to the one described in Sec. 3.3.1. In this case, agents with too high prices will sell exactly nothing at all. If those prices are high enough, production will stay at zero, even if the price is lowered. That means that trial-and-error price adaptation does not work here either, but the result of the too high price is not a constant increase (as in Sec. 3.3.1) but a random walk of the price: Every new price is accepted because the utility does not change (it stays at zero). Like this, it is always possible that the price will eventually become low enough so that the agent finds the signal that she should decrease her price again.

Here, the solution to avoid these random walkers is that every agent that has production zero must enter testing mode immediately (i.e. she will enter testing mode for sure in the beginning of the next testing cycle) and test a lower price. Like this, no agent will ever be in a position where her price is so much higher than the average that a decrease of the price does not increase her utility.

Actually, this effect is very important in general for these types of simulations: Since the agents essentially do hill climbing, there needs to be a slope in order to find the uphill direction.

The following tables summarize the results for $N = 10$:

quantity	large N pred	xcalc sim	tr-and-err sim
p	1.44	1.49	1.54
q	0.692	0.673	0.648
U	0.399	0.394	0.386

(3.12)

and $N = 50$:

quantity	large N pred	xcalc sim	tr-and-err sim
p	1.08	1.07	1.13
q	0.925	0.939	0.887
U	0.48	0.429	0.400

(3.13)

A comment to the result for the equilibrium utility: In the simulation with 50 agents the experimental result seems far off from the analytical prediction. This is easy to explain if one looks at Fig. 3.2. All measured quantities except utility show approximately as strong upward as downward fluctuations. These fluctuations are caused by testing of "bad" prices. For utility there can be only downward fluctuations as it is utility itself that determines whether a tested price is "good" or "bad". By definition,

in equilibrium all agents will always go back to their former price after testing. Now let U_{SS} the utility level of the agents that are in steady state but do currently not test a new price. If the price testers are included in the averageing, average utility in equilibrium will always be smaller than U_{SS}. If the downward fluctuations are small (as they are for square root consumption utility) the error will be small. Here, the fluctuations are very strong, as the agents follow logistic consumption behaviour, and U_{SS} should thus not be measured by the average utility. A better value for U_{SS} can be obtained if one takes the maximum of utility of a single arbitrary agent. This would be 0.45 for the trial-and-error simulation with 50 agents and 0.49 in case of calculated demands.
For simulations with more agents, the problems with measurement will be worse, and we chose to exclude the price testers before averageing, see below.

Still, the result for the trial-and-error simulation is far off from the "calc" simulation. A closer result would probably be obtained if T_p would be chosen even higher than 500000 (or equivalently if the fluctuations would be reduced). Furthermore, simulation time should be increased, too, in order to get a more precise result, but the simulation takes so long (even 5 times longer than the corresponding simulation in chapter 2, because of the increase of T_p) that this makes little sense. The best thing to do is to take the result of the corresponding "calc" simulation. It is interesting to note here that the equilibrium price is *lower* than predicted. This effect is analysed below.

As in Sec. 2.6 we can also consider to simulate systems with more agents. For simulations with large N ($N >\approx 100$), a third change (compared to the corresponding simulations with square root utility) has to be introduced. It merely has to do with the method we use to measure the resulting economic quantities in equilibrium: For large N, the fluctuations become so strong that there is an asymmetry between upward and downward fluctuations, because money and production cannot fluctuate downwards below zero while there is no bound in the upward direction. As soon as this asymmetry occurs, the equilibrium values of the economic quantities become dependent on how much the prices are allowed to fluctuate. For example, the larger the prices are allowed to fluctuate, the larger will the upward fluctuation of production be in case of a testing of a lower price. This means that the agents that are in equilibrium (the ones that currently do not test prices) would give ever more money to the ones that test lower prices while they always give nothing to the ones that test higher prices. Like this, the average amount of money M_{SS} of these agents is dependent on the amplitude of the price fluctuations: M_{SS} decrease as the amplitude of the price fluctuations increases. But as the Nash equilibrium price p_{SS} itself is proportional to M_{SS}, even p_{SS} will depend on this amplitude. Utility in equilibrium, as a special case, can fluctuate only downwards, as every test of a new price must lead to a worse utility. If the amplitude of the price fluctuations is high, utility will fluctuate to zero in case of a test of a higher price. The fluctuations due to testing of lower prices will go much below zero (as the production strongly fluctuates upwards) and are again dependent on how much the prices are allowed to fluctuate.

Here, as we do not vary the amplitude of the price fluctuations, the important observation is that this effect will appear when we are considering systems with large N: As is shown in the next section, the fluctuations in *production* will be proportional to N. With our choice of the maximum fluctuation amplitude of the prices given by $p \to 1.1p$ or $p \to p/1.1$ the effect appears for $N >\approx 100$ as can be seen in the tabular below.

quantity	large N pred	xcalc sim
p	1.04	0.97
U	0.49	0.32
q	0.96	1.03
M	1	1

(3.14)

For the first time, the experimental value for p_{SS} is considerably lower than the analytical prediction $p_{analyt} = \frac{N+3}{N-1}M$. As seen above, this is because we have to insert M_{SS} into the formula for p_{analyt} and not $M_{tot}/N = 1$ as we did up to now. The measured equilibrium utility is much below the predicted value, because of the strong downward fluctuations of individual utility. The same applies to production: Its measured equilibrium value is higher than predicted because of the strong upward fluctuations of the production of the agents that test lower prices.

For even larger N, the experimental values would differ even more from the predicted values. For example the average measured utility will become negative while the analytical solution says it will converge to 0.5.

For the measurement of the equilibrium values it seems most natural to exclude always the agents that are currently testing a new price before averageing. At the same time we can include the dependence on the steady state price on the fluctuations in the analytical solution. This is shown in the next section. Like this, measured and predicted values for the case $N = 100$ are given by:

quantity	large N pred	xcalc sim
p	0.99	0.98
U	0.49	0.49
q	0.96	0.97
M	0.95	0.95

(3.15)

The agreement is much better now. Note especially, that the average amount of money of the agents in equilibrium is no longer fixed at 1. It is below below 1 because the "low-price-testers" posses an disproportionate share of M_{tot}. Note also, that the predicted values for utility and production did not change compared to the last tabular. This is because the refined analytical solution merely includes the calculation of M_{SS}. The predicted value of the steady state price is then obtained by inserting M_{SS} instead of M into the formula for p_{analyt}. $q = M/p$, $x = q/(N-1)$ and therefore also the prediction for U is thus not changed.

3.4 Fluctuations

A striking feature of the model are the observed strong fluctuations. Indeed, one expects the fluctuations in this model to be much larger than in the first model, since in the $x \to 0$ limit (or equivalently the $N \to \infty$ limit) the slope of the utility function is one and not infinity, which means that substitution is always at hand. This fact is illustrated in Fig. 3.11, where fluctuations in individual production are compared between the two models for $N = 100$. The very different kind of consumption utility makes the fluctuations increase with N much stronger than in the first model: Compare the fluctuations in the system of 100 agents to the system of 10 agents (Figs. 3.9 and 3.10). For logistic consumption utility both absolute as well as relative fluctuatuations in production are proportional to N: If in the homogenous situation one agent changes her price by δp, her production will fluctuate by approximately

$$\delta q \approx 0.5\,\delta p \times N. \tag{3.16}$$

Note, that is a striking difference to the square root model, where the fluctuations increase (in absolute value) only $\propto N^{1/3}$.

The calculation that leads to the result Eq. 3.16 is the same as in Sec. 2.7: First calculate the allocation of money in case all agents except agent i charge price p. This reduces to solving the following linear system of equations:

$$M_{oth} = p((N-2)x_{oth} + x_{i,oth}) \tag{3.17}$$

$$M_i = M_{tot} - (N-1)M_{oth}. \tag{3.18}$$

Here, x_{oth} is the amount of goods that the "other" agents exchange among themselves and $x_{i,oth}$ is the amount of goods that agent i buys from an arbitrary other agent. Both x_{oth} and $x_{i,oth}$ are given by Eq. 3.3.

After solving for M_{oth}, one can calculate $x_{oth,i}$ (the amount of goods that the other agents buy from agent i) by inserting the result for M_{oth} into Eq. 3.3. The production $q_i = (N-1)x_{oth,i}$ is then given (to first order in $\delta p \equiv p_i - p$) by:

$$q_i = q_{hom} - \frac{1}{2}\frac{pN^2 - (3p+2M)N + 2p + 8M}{p^2 N}\delta p. \tag{3.19}$$

Assuming that the system is in the relaxed state, we can set $p \approx 1$. In the limit $N \to \infty$ the fluctuation in production is then given by Eq. 3.16. Note also that Eq. 3.16 gives both absolute as well as relative fluctuations at the same time, as in the limit $N \to \infty$ $q_{hom} \equiv M/p \approx 1$.

Thus, one sees that in case of a lowering of the price, there will be very strong upward fluctuations in production if there are many agents and in case of an increase in price the production can drop to zero. This is not possible for the square root utility where production fluctuates not more than about 20% up or down (see Sec. 2.7).

A consequence of the strong fluctuations is that (compared to the first model) the average utility in equilibrium is increasing with N only for small N: For large N it is again decreasing and eventually even becomes negative: As with production, utility shows two kinds of fluctuations. If an agent increases her price, utility will drop to zero along with the production, but in case of a lowering of the price, utility can fluctuate much below zero as the production goes up! It is these very strong downward fluctuations in the individual utility which make the average utility useless as a measure of U_{SS}, the utility level of the agents in equilibrium. The *individual* utility is (apart from the downward fluctuations) positive and exhibits a strong cutoff in the upward direction, as after testing of a new price in equilibrium (leading eventually to a decrease in utility) utility will jump back to its former value. A measure of U_{SS} can be obtained by excluding all agents that are testing a new price. Now, we again observe that utility is increasing with N as predicted by Eq. 3.7, see Fig. 3.12. As explained in the last section, we adopt the same procedure for the measurement of all economic quantities.

A direct consequence of the *asymmetry* of the fluctuations is that M_{SS} is $< M_{tot}/N$ for large N (see Fig. 3.13). This has an important influence on the Nash equilibrium price, as p_{SS} is proportional to M_{SS}: For $N >\approx 100$, the average equilibrium price goes below 1 as is shown in Fig. 3.14. Analytically, this can be understood because the M in Eq. 3.5 must be replaced by M_{SS}. Like this p_{SS} becomes dependent on the fluctuations.

An analytical expression for M_{SS} can be found like that: We introduce the parameters r and f that describe the price fluctuation. $r \times N$ is the average number of agents that test lower prices (of course it is likewise the average number of agents that test higher prices). On the other hand, $p_l \equiv f \times p_{SS}$ is the expectation value for a lower price which is tested. The values of r and f must be determined very exactly, as the result will be very sensible to slight changes in r and f. The values have already been determined below Eq. 2.35, they are $r = 0.04545$ and $f = 0.9531$.

For the calculation we assume that the rN agents that test a higher price can be excluded from the system as for these $M \approx 0$. This leaves rN agents with price p_l and money M_l and $(1-2r)N$ agents with price p_{SS} and money M_{SS}. The usual steady state condition is

$$M_{SS} = p_{SS}\{((1-2r)N-1)x_{SS} + rNx_{SS,l}\} \tag{3.20}$$

$$M_l = \frac{M_{tot}/N - (1-2r)M_{SS}}{r} \tag{3.21}$$

where for x_{SS} and $x_{SS,l}$ Eq. 3.3 has to be used. Before solving for M_{SS}, we insert also $p_l = fp_{SS}$ and finally Eq. 3.5, $p_{SS} = \frac{(N+3)M_{SS}}{N-1}$. The resulting system of equations is still linear in M_{SS} and yields a quite complicated formula for the dependence of M_{SS} on $(N, M \equiv M_{tot}/N, r, f)$. Because all parameters except N are constants, we insert these values and display M_{SS} as a function of N alone:

$$M_{SS}(N) = 2\frac{(N-1)(0.9504N-1)}{0.001843N^3 + 1.806N^2 - 3.713N - 1.905} \tag{3.22}$$

The function is plotted in Fig 3.15.

By reinsertion of M_{SS} into the expression for p_{SS} the refined formula for the steady state price is found.

The following tabular shows the result of an experimental test of the new formulae for M_{SS} and p_{SS}. In the system of 300 agents we find:

quantity	large N pred	xcalc sim
p	0.815	0.803
M	0.804	0.803

(3.23)

In order to get some more insight into the dependence of M_{SS} on r and f, it would be nice to find an approximation to the full expression. The dependence on N is $\propto 1/N$ for large N, but because of the small prefactor of the N^3 term in the denominator, the $1/N$ term dominates only for very large N ($N >\approx 5000$). For the tayler expansion in the interesting range $N < 1000$ many terms must be considered and thus the result does not become clearer.

3.5 Some discussion

As shown in the previous sections, the large fluctuations in a model with logistic consumption behaviour can result in a lot of (initially not expected) new effects. In terms of stability, these fluctuations are hard to handle. We tried to change as little as possible in the code when we switched from square root to logistic consumption utility so that it was possible to compare the results of the two models, but in a possible further development of a model with logistic consumption utility one should nevertheless look after a way to reduce the fluctuations. Below are some ideas how to do that:

The easiest way to control the fluctuations is by adapting the amplitude of the allowed fluctuations in the prices. The amplitude should be reduced when large N are considered. This does not necessarily mean that the approach to the steady state will take longer: If the system starts at a too high price level and the price adaption amplitude is very high, large price decreases will not be accepted (even if the new price is above p_{SS}) because the resulting increase in production (work) is so high.

The strong fluctuations can also be seen as a result of too much information and flexibility. For example, introducing a spatial component will reduce these fluctuations: In case of a small price change, only the consumers being located right at the borderline between two suppliers will switch.

Similarly, if consumers reacted only imperfectly to price differences, the strong fluctuations would go away. This could for example be achieved by using something like

$$\frac{x_{ij}}{x_{ik}} \propto \frac{e^{-\beta p_j}}{e^{-\beta p_k}} \tag{3.24}$$

for the allocation of consumption when price differentials are small.[1] Such an approach would be consistent with approaches in Statistical Physics.

[1] In fact, the leading terms for consumption are

$$x \sim (P - p_j) + \frac{1}{N} \; . \tag{3.25}$$

One could replace this by

$$x \sim (P - p_j)\,|\tanh[\beta(P - p_j)]| + \frac{1}{N} \; . \tag{3.26}$$

For $\beta \to \infty$ this would return to what we had before (zero noise limit); for $\beta \to 0$ this would switch off the effect of the first term completely (large noise limit).

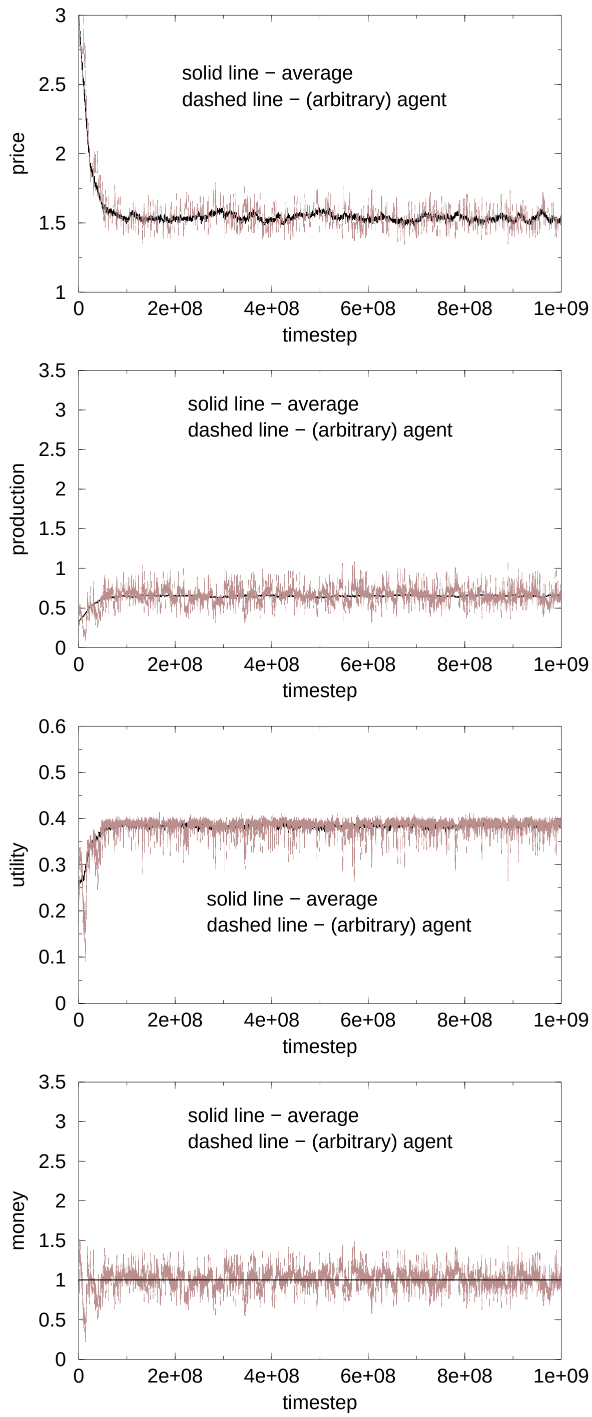

Figure 3.1: *Relaxation for a logistic consumption utility. System with* $N = 10$ *agents.*

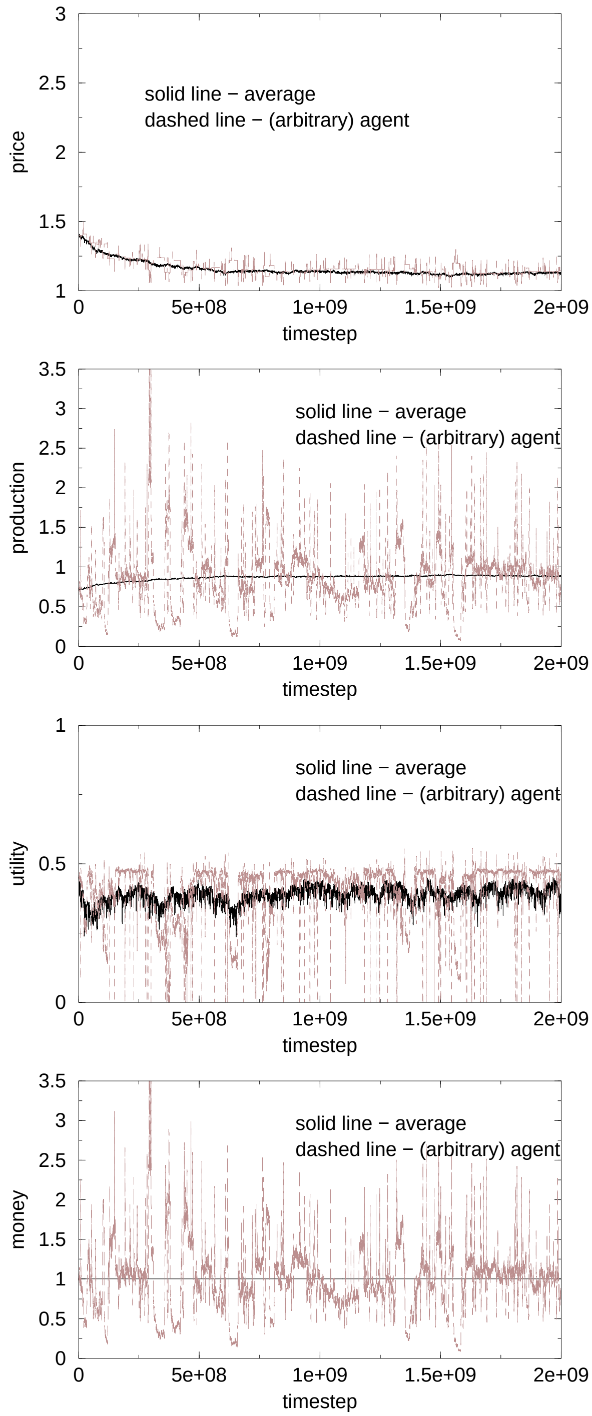

Figure 3.2: *Relaxation for a logistic consumption utility. System with $N = 50$ agents. Because of the large jumps of utility after price changes we need $T_p = 500000$ to get a reasonable result for the equilibrium price level. Here, the downward fluctuations in individual utility due to a testing of a lower price can be quite strong. For example the fluctuation at timestep $3e08$ goes down to about -5. Here, one can see very clearly how every test of a lower price is accompanied by a large upward fluctuation of money and production, and by a large downward fluctuation of utility.*

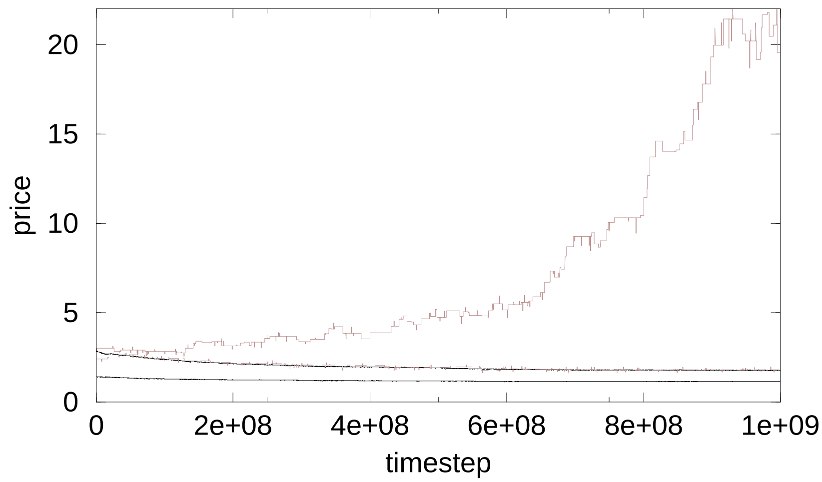

Figure 3.3: *If "low-producers" are not forced to lower the price, it is possible that these agents get decoupled from the system. The upper solid line shows the result for the average price in a simulation with* 50 *agents where nothing is done to avoid this loss of agents. The thin line is the price history of one agent that happened to get decoupled. For comparision, the lower solid line shows the average price from Fig. 3.2 where there is no loss of agents. Clearly, the system relaxes to a lower steady state price, then.*

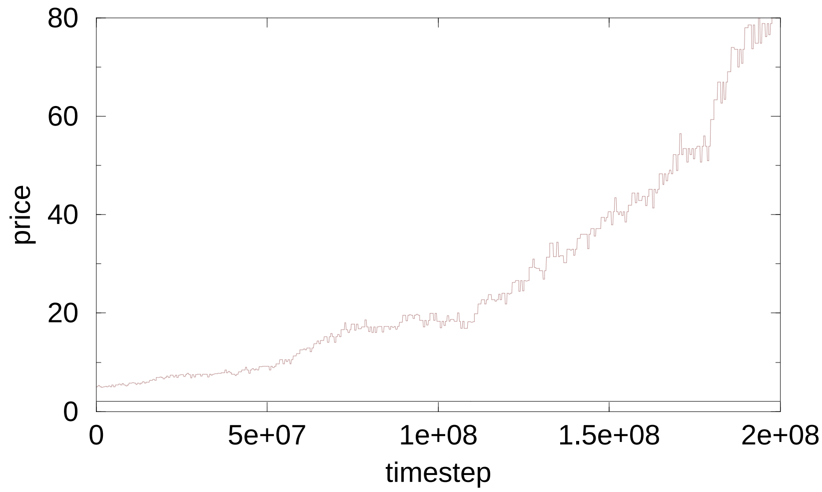

Figure 3.4: *Simulation where all agents have a fixed price* $p = 2$ *and one starts at* $p_h = 5$ *and is allowed to adapt her price. This agent gets a wrong signal to where she should adapt the price.* $N = 50$.

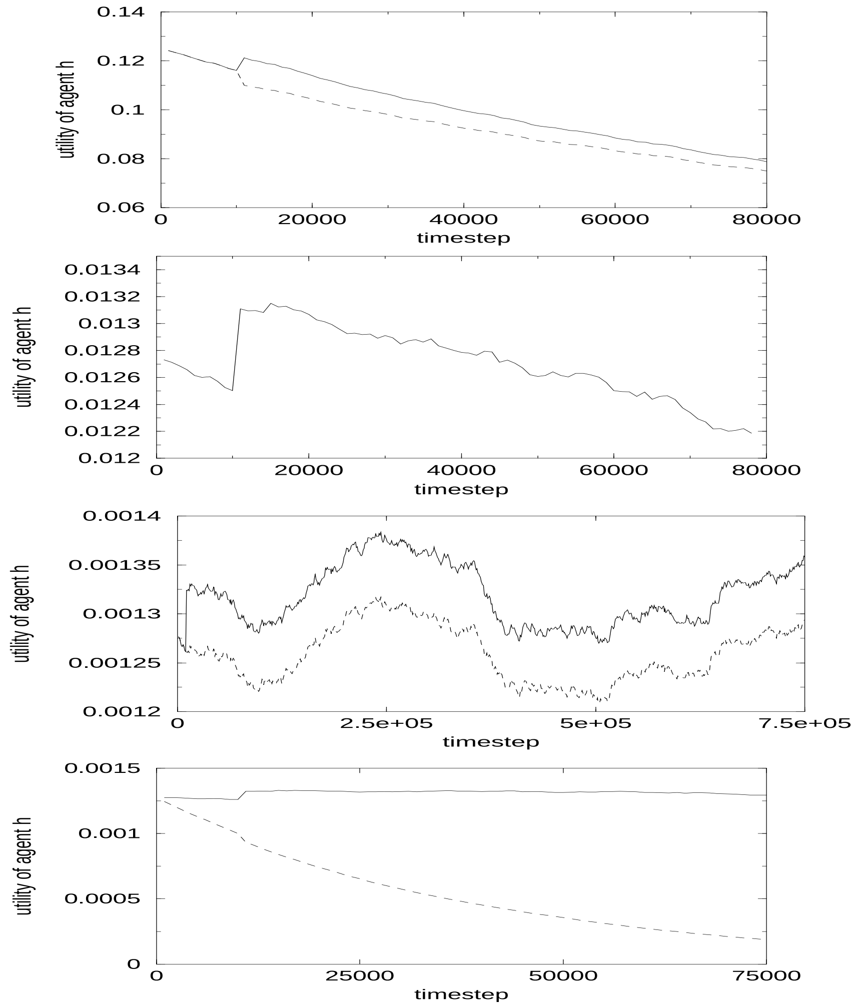

Figure 3.5: *The reason why agents with too high prices adapt their prices in the wrong direction. Simulation where one agent h starts at a too high prise of $p_h = 5$ and all the others have a fixed price of $p = 2$. The optimal demand of the expensive product would be zero, but the decrease of it is slowed down ever more by the noise in the system. The effect of a change of p_h with $f_{p,rand} = 1.05$ in timestep $t = 10000$ is shown for different initial values of the demand for that product. TOP: The initial demand $x_{h,0}$ of the expensive product is about one tenth of the demand $x_{oth} \approx M/p/(N-2)$ for the other products. The signal to reduce x_h is very strong and neither a price increase (solid line) nor a price decrease (dashed line) can stop the fall of the utility of agent h. SECOND: Same simulation, but with $x_{h,0} \approx 1/100 x_{oth}$ and only the price increase is shown. Because, as $x_h \to 0$, the noise in the system slows down the decrease of x_h ever more, it takes longer for the transient jump in utility to die out. THIRD: As $x_{h,0} \approx 1/1000 x_{oth}$, the agent is essentially decoupled from the system. A price increase (solid line) merely shifts the utility up and is therefore likely to be accepted. Dashed line: no price change. BOTTOM: solid line: same as THIRD, the dashed line is also the same, but without noise. That means that the agents can only adapt x_h. The price increase would not be accepted, then, as x_h (and along with it utility) decays exponentially to zero without any damping.*

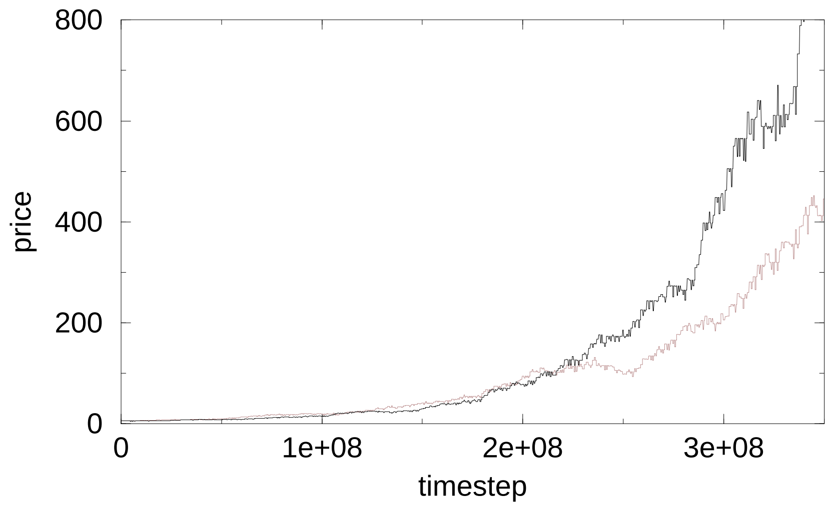

Figure 3.6: *The thin line is the same as in Fig. 3.4, for the solid line,* T_p *has been changed to* 300000 *instead of* 500000*. The* t*-values of this curve are scaled by a factor* $5/3$*. Clearly, the inflation of* p_h *per* T_p *is stronger when* T_p *is reduced, i.e. the "wrong-signal" effect is stronger for smaller* T_p*.*

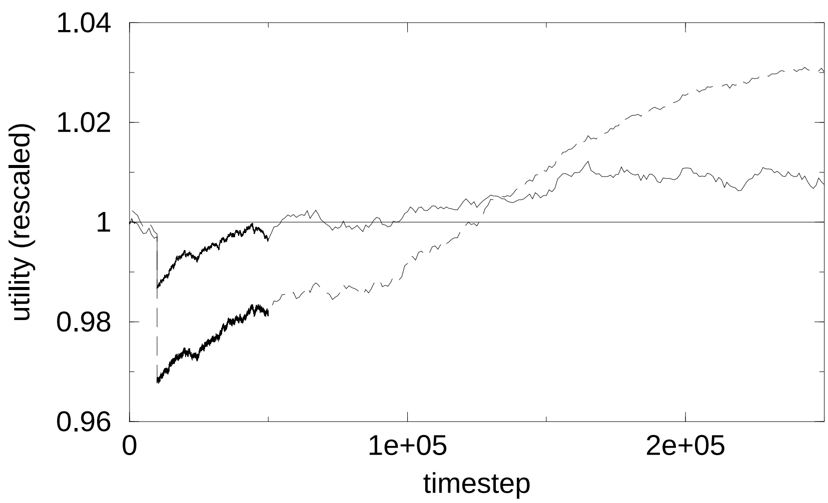

Figure 3.7: *The recovery time after a change of price is longer for logistic consumption utility than for square root consumption utility because in case of logistic consumption the jump in utility after a price change is about* 4 *times larger. Both simulations with* $N = 50$ *agents. The prices are fixed at about* 40% *above the respective analytically predicted equilibrium value (i.e.* $p_{sqrt} \approx 0.6$ *and* $p_{log} \approx 1.4$*) and at time* $t = 10000$ *one agent changes her price according to* $p \to p_i = p/1.02$*. The utility of this agent is shown, rescaled so that the average utility in the system would be* 1*. Solid line: square root consumption utility, dashed line: logistic consumption utility.*

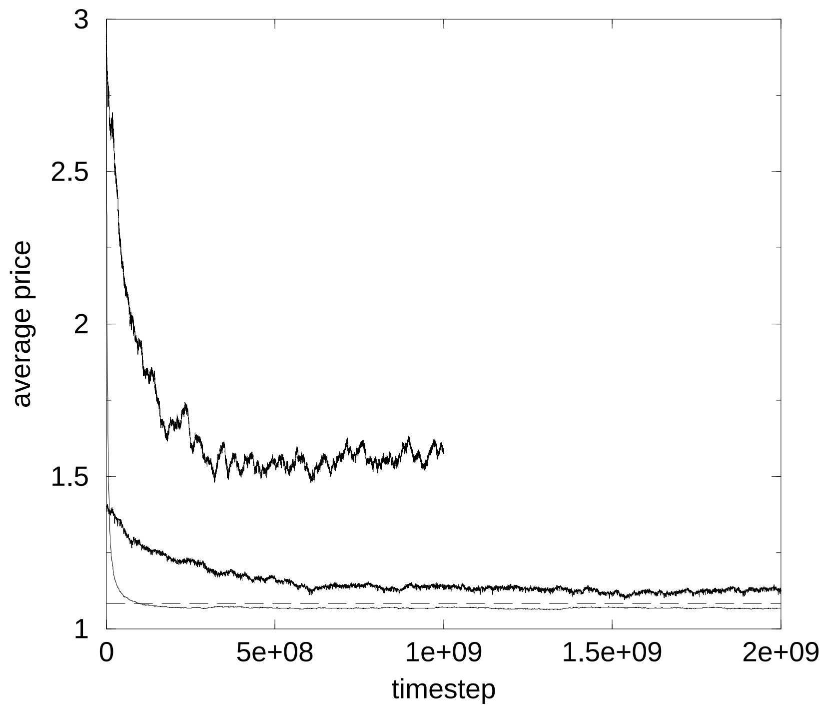

Figure 3.8: $T_p = 100000$ *is not enough for trial-and-error adaption of demands in a system of* 50 *agents. The curves from top to bottom: Simulation with* $T_p = 100000$*; simulation with* $T_p = 500000$*; analytical result from Eq. 3.5 with* $M = 1$ *(dashed line); result of the corresponding simulation with calculated adaption of demands (with scaled* t*-values). Clearly, the equilibrium result of the "trial-and-error simulation"is much closer to the result of the "calc simulation," if* $T_p = 500000$ *is used. – The reason why the equilibrium price in the calc simulation is below the analytical value is explained in Sec. 3.4.*

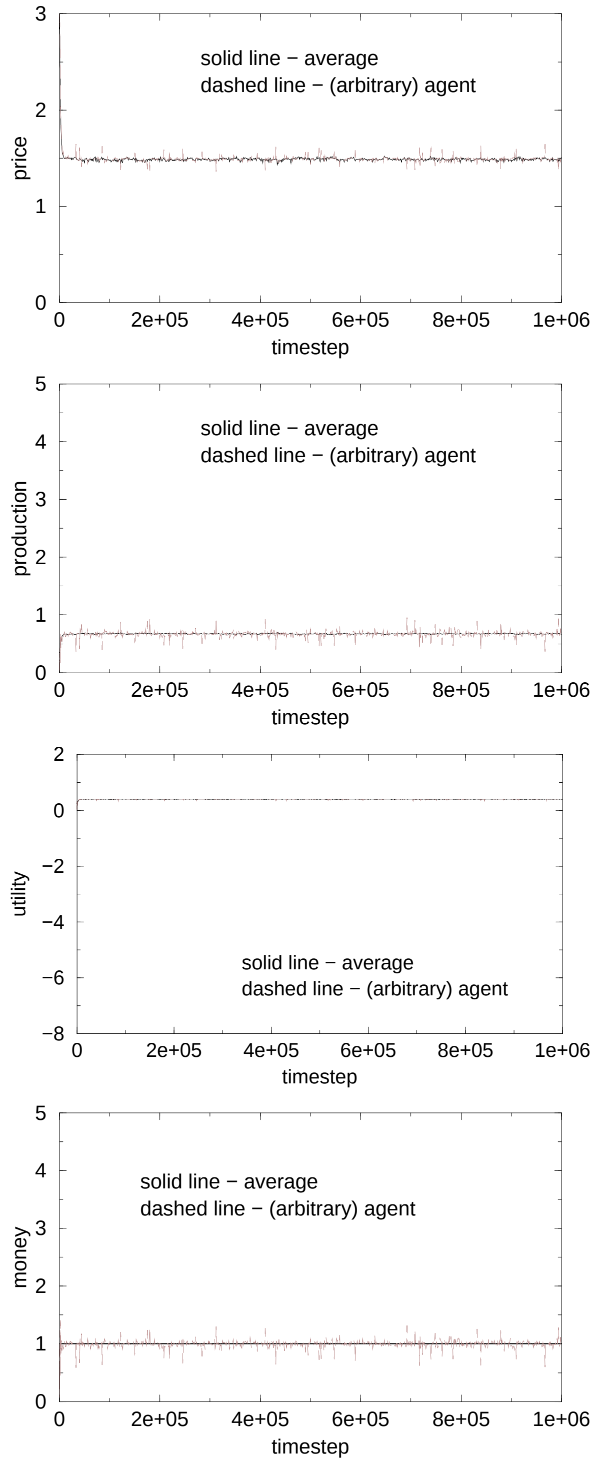

Figure 3.9: *Relaxation for a logistic consumption utility when demands are calculated. System with* $N = 10$ *agents.*

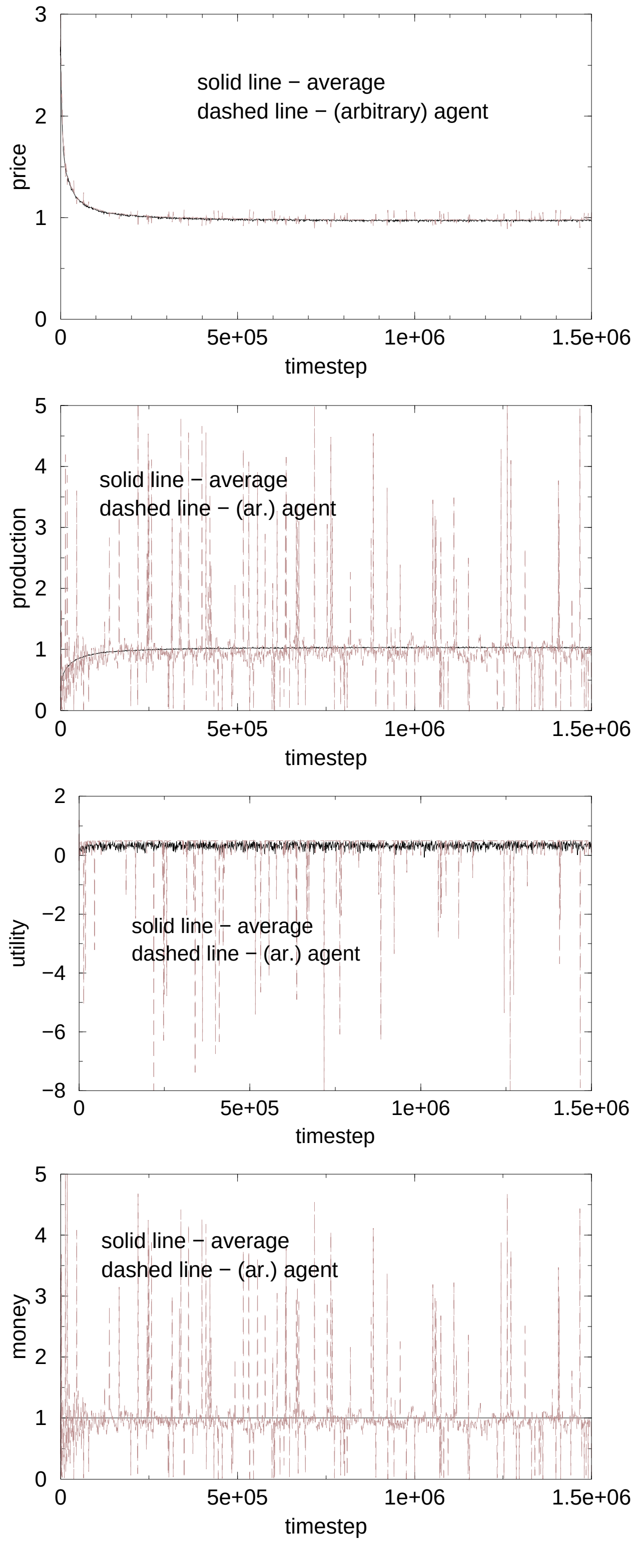

Figure 3.10: *Relaxation for a logistic consumption utility when demands are calculated. System with* $N = 100$ *agents.*

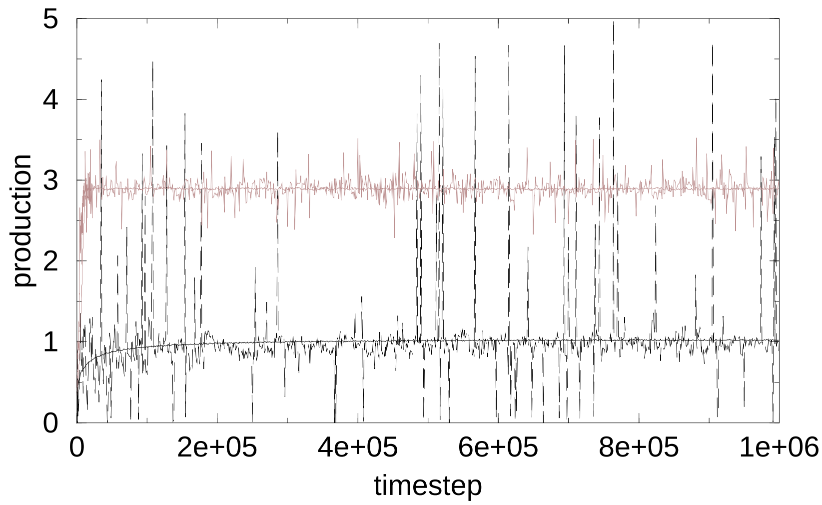

Figure 3.11: *Fluctuations in the model with logistic utility function are much larger than in the basic model: Top curves: basic model, bottom curves: logistic model. Always average production as well as the production of a single (arbitrary) agent is plotted. Simulations with* 100 *agents.*

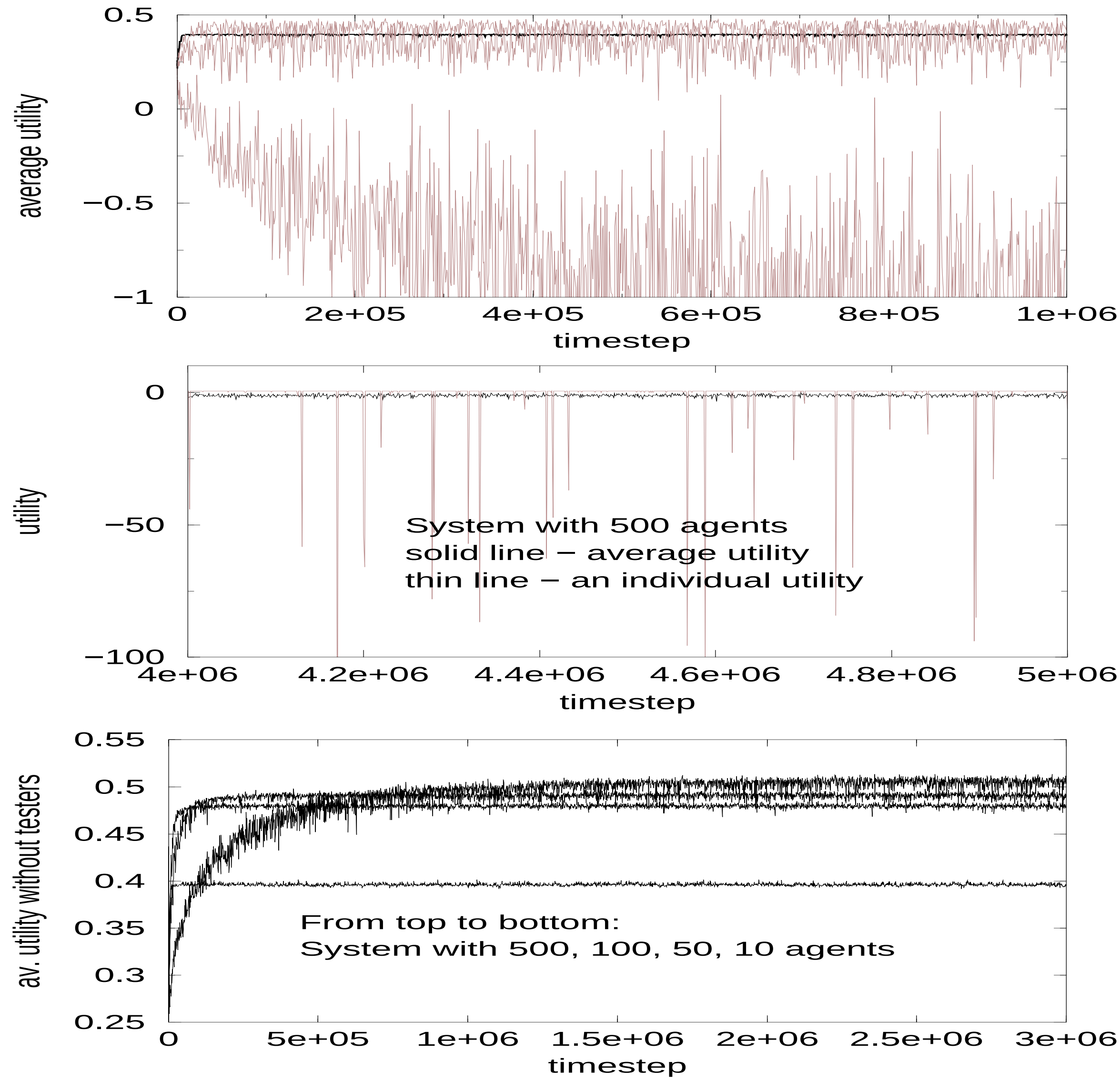

Figure 3.12: *TOP: average equilibrium utility is decreasing with N for large N: The black line is the average utility in the system of 10 agents. The topmost thin line is for the system of 50 agents, the lower thin lines for systems of 100 and 500 agents, respectively. The reason for the decrease in the large N limit is the large fluctuations, as can be seen in the MIDDLE figure: In the system of 500 agents, the utility of a single agent is usually positive, but exhibits very strong fluctuations in direction of negative utility. Upwards there is a strong cutoff, which is (in contrast to average utility) increasing with N. BOTTOM figure: In order to measure the equilibrium utility one must average only over the utility of the agents that do not test new prices. Equilibrium utility is then increasing with N as predicted by Eq. 3.7.*

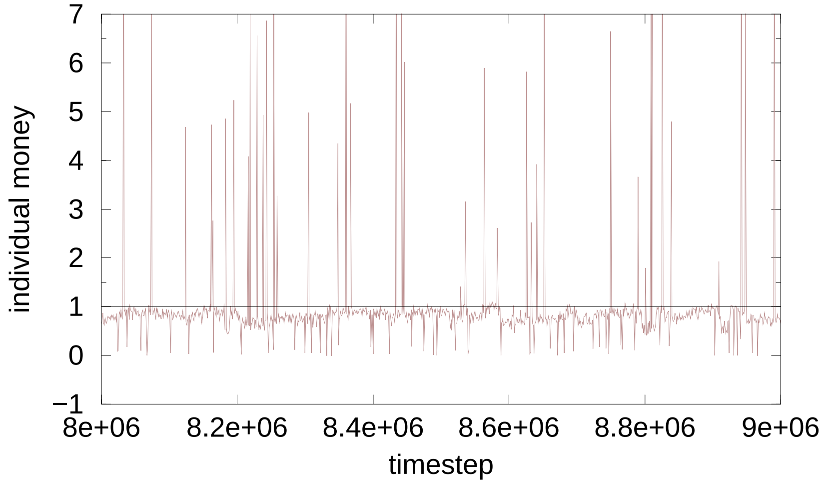

Figure 3.13: *In the simulation with* 300 *agents,* M_{SS} *is smaller than* $M_{tot}/N = 1$*. Obviously, the reason is the asymmetric fluctuations in individual money: Consider the curve for the individual money above. The agent is in equilibrium except when she tests a new price. If she changes her price to* p_{test}*, her production will fluctuate by* $\delta q = 0.5N(p_{test} - p_{SS})$*. Her money thus fluctuates by* $\delta M = p_{test} \times \delta q$*. Inserting some values: For* $N = 300$*,* p_{SS} *is about* 0.8*. Thus, for an average price change where* $p_{test} \approx (1 \pm 0.05)p_{SS}$*,* δM *is predicted to be about* ∓ 5*. This is confirmed by the figure in case of price decreases. For price increases it means that the money will drop to zero as the agent will not sell anything. – The conclusion is that the agents that test lower prices will share a disproportionate share of the total amount of money in the system and thus* M_{SS} *becomes smaller than* M_{tot}/N*. Clearly, the effect becomes stronger as* N *increases.*

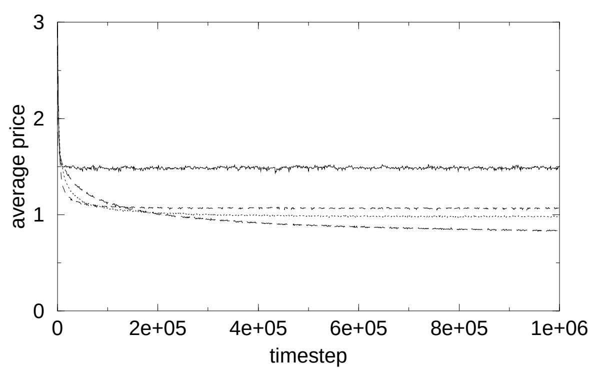

Figure 3.14: *Average equilibrium price is decreasing with* N*: From top to bottom: Simulations with* 10*,* 50*,* 100 *and* 500 *agents. For* $N = 100$ *and* 500 *the price is below* 1 *which is a consequence of the fact that* $p_{SS} \propto M_{SS}$*, but* $M_{SS} < 1$ *for large* N*.*

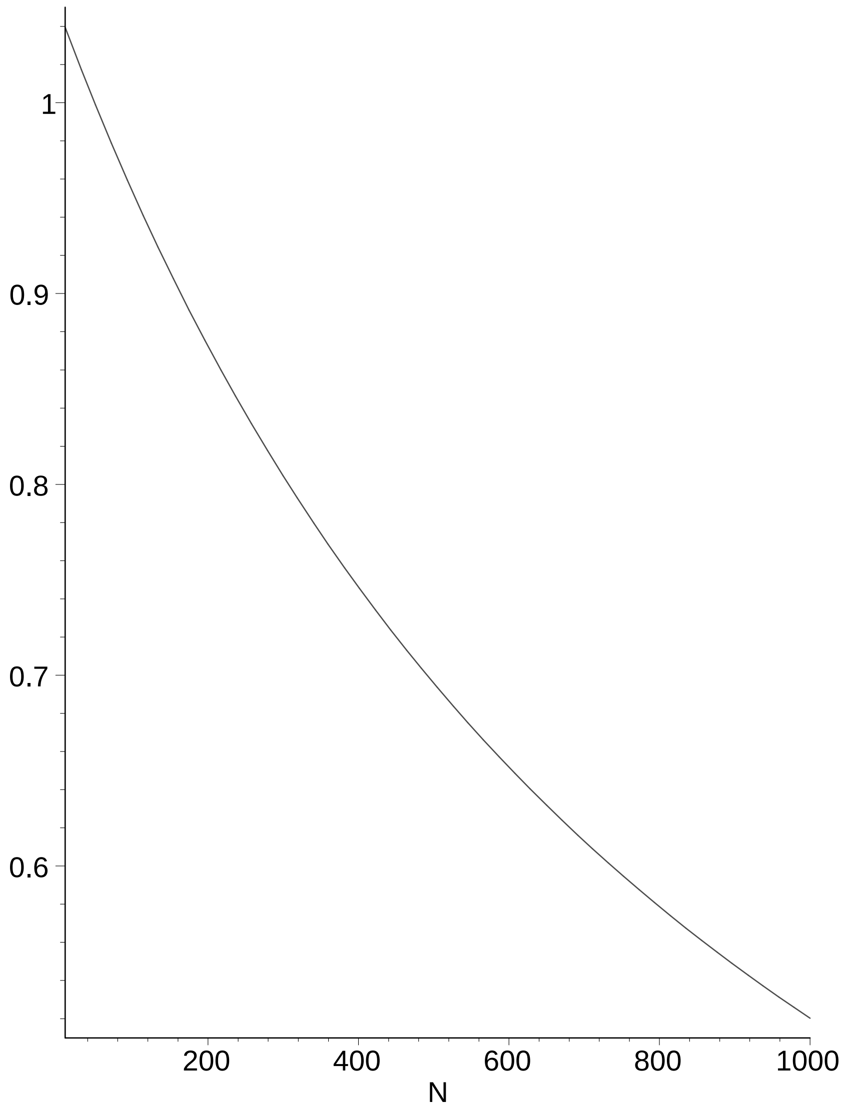

Figure 3.15: M_{SS} *as a function of* N*, for* N *in the interval* $[10, 1000]$ *(plot of Eq. 3.22).*

Chapter 4

Summary

We have presented a simple dynamic model of a market. Certain versions of the model can be treated analytically. Simulation offers the possibility to go beyond the analytically solvable cases. In both cases, for stable solutions it is crucial to select the dynamics correctly. In the model of this paper, price adaptation has to happen on a much slower time scale than consumption adaptation, otherwise price evolution will not behave reasonably. This is intuitively plausible; nevertheless, it needs to be taken into account both when building simulations models and possibly when regulating the real world.

Acknowledgments

The author would like to thank the following persons for their various kinds of assistance during the work for this diploma thesis:

Prof. Kai Nagel, my supervisor, who laid the foundation of the model and whose experience in the field of agent-based computer simulations often was amazing and always of invaluable help.

Prof. Matthias Troyer, my computational physics teacher, who initiated my fascination for computer simulations in his very work intensive RGP lectures.

My officemate Christopher Kayatz who wrote his diploma thesis during the same time. Many times when everything seemed to go wrong with the behaviour of my agents it was of inestimable value to be able to talk to someone who knows that problem. Behind our closed office doors we shared a lot of good music that no one else in the department would like to listen to.

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Last but not least, my girlfriend Mari for the love she gives to me and her patience in the long times when she wouldn’t hear anything from me because my agents kept me occupied...

Appendix A

A.1 Analytical solutions for the square root utility

A.1.1 Optimal prices

Given that one knows the consumer's reaction Eqs. 2.7, one can calculate the optimal price. For this, one has to replace q_i by $\sum_{k \neq i} x_{ij}$, leading to

$$\tilde{U}_i = -\frac{1}{2p_i^4} \left(\sum_{j \neq i} \frac{M_{j,t-1}}{\sum_{k \neq j} \frac{1}{p_k}} \right)^2 + 2 \sum_{j \neq i} \sqrt{x_{ij}} + \lambda_i \left(\frac{1}{p_i} \sum_{j \neq i} \frac{M_{j,t-1}}{\sum_{k \neq j} \frac{1}{p_k}} - \sum_{j \neq i} p_j \, x_{ij} \right) . \tag{A.1}$$

Note that λ_i does not depend on p_i.

Two limiting cases are easy to calculate:

- $N \to \infty$. In this case, the dependence of $\sum_{k \neq j} 1/p_k$ on p_i vanishes and thus all λ_j become independent of p_i. In this case, the derivative of $\tilde{U}_i$ w.r.t. p_i becomes

  $$2p_i^{-5} \left(\sum_{j \neq i} \frac{1}{\lambda_j^2} \right)^2 - \lambda_i \, p_i^{-2} \sum_{j \neq i} \frac{1}{\lambda_j^2} . \tag{A.2}$$

  Solving for p_i results in

  $$p_i^3 = 2 \, \lambda_i^{-1} \sum_{j \neq i} \frac{1}{\lambda_j^2} . \tag{A.3}$$

  In the homogeneous case, $\lambda_i^2 = \lambda_j^2 = \lambda^2 = (N-1)/Mp$, and therefore

  $$p = \frac{2^{2/3} \, M}{(N-1)^{1/3}} . \tag{A.4}$$

  This yields

  $$x = 2^{-2/3} \, (N-1)^{-2/3} \ \text{ and } \ q = 2^{-2/3} \, (N-1)^{1/3} . \tag{A.5}$$

- $N = 2$. In this case, $\sum_{k \neq j} 1/p_k = 1/p_i$. In this case,

  $$\frac{\partial \tilde{U}_i}{\partial p_i} = p_i^{-3} , \tag{A.6}$$

and setting this equal to zero means $p_i = \infty$. This may look surprising at first, but makes sense since this is the monopoly situation: Each agent buys only one good, and so the seller can raise prices without bound and still make the same amount of money.

The exact approach to find the optimal price which agent i should charge if she knows all other prices must include a correct treatment of a change in the allocation of money that results from a change of the prices. In the simulation where demands are calculated this is the only mechanism that delays the immediate adaption of demands to a new price situation. So in order to study the effect which a setting of a new price has, one first has to calculate the resulting allocation of money. This gives (by Eq. 2.7) directly the new demands of all agents and therefore also the resulting change in utility.

The allocation of money that results from a given price distribution must fulfill the steady state condition

$$M_i = p_i \sum_{j \neq i} x_{ji}, \tag{A.7}$$

where

$$x_{ji} = \frac{M_j}{p_i^2 \sum_{k \neq j} \frac{1}{p_k}}. \tag{A.8}$$

This is a linear system of equations for the M_i's as functions of the prices. Since $\sum_{j \neq i} x_{ij} = M_i$ (meaning that every agent spends all her money in every timestep) the total amount of money is conserved and therefore only $N-1$ of the M_i's are independent. The determinant is no longer zero, though, if the equation for M_N is replaced by

$$\sum_{i=1}^{N} M_i = M_{tot}, \tag{A.9}$$

thus introducing the conserved parameter M_{tot}. The result is then:

$$M_i = \frac{M_{tot}}{2} \frac{\sum_{j_1 < j_2 < \ldots < j_{N-2}, j_k \neq i} p_{j_1} p_{j_2} \ldots p_{j_{N-2}}}{\sum_{j_1 < j_2 < \ldots < j_{N-2}} p_{j_1} p_{j_2} \ldots p_{j_{N-2}}}. \tag{A.10}$$

Assuming that all agents keep their prices fixed, the above formula gives the allocation of money as function of the price which agent i sets. This means that agent i also knows the resulting demands which puts her utility function in the form

$$U_i(p_1, \ldots, p_N) = -\frac{1}{2} q_i(p_1, \ldots, p_N)^2 + 2 \sum_{j \neq i} \sqrt{x_{ij}(p_1, \ldots, p_N)}. \tag{A.11}$$

In this form utility can be maximized without any constraint to find $p_{i,opt}(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots p_N)$, but even for $N = 3$ the derivative w.r.t. p_i becomes so complicated that the roots cannot be found analytically.

So let us as a first step assume that all other agent's prices are fixed and identical. According to the above formula the allocation of money will be

$$M_i = M_{tot} \frac{p}{2p + (N-2)p_i} \tag{A.12}$$

for agent i and

$$M \equiv \frac{M_{tot} - M_i}{N-1} = \frac{M_{tot}}{N-1} \frac{p + (N-2)p_i}{2p + (N-2)p_i} \tag{A.13}$$

for all other agents. Using these expressions to calculate the demands, Eq. A.11 gives the following expression for the utility:

$$U_i = -\frac{1}{2}\frac{M_{tot}^2(p+(N-2)p_i)^2}{p_i^4(2p+(N-2)p_i)^2\left(\frac{N-2}{p}+\frac{1}{p_i}\right)^2} + 2\sqrt{\frac{(N-1)M_{tot}}{2p+(N-2)p_i}} \tag{A.14}$$

After derivation w.r.t. p_i and setting to 0, the optimal p_i would still be among the roots of an expression which is too complicated to treat exactly. It is interesting, though, to explore graphically how agent i would set her price for different price levels of all the other agents. This is shown in Fig. A.1: If all agents charge a price p which is larger than $p_{Nasheq} \approx 2^{2/3}M/(N-1)^{1/3}$, agent i should set her price optimally somewhere *between* p_{Nasheq} and p, not *below* p_{eq}. Accordingly if $p < p_{Nasheq}$, e.g. if p is chosen to be the “system-wide” equilibrium price $p_{SO} \equiv M/(N-1)^{1/3}$, p_i will optmally be between p and p_{Nasheq}, which means that the system-wide optimal price is instable and the system will evolve towards the Nash equilibrium price. Of course, as $p \to p_{Nasheq}$ we observe that $p_i \to p_{Nasheq}$, too.

A.2 Analytical solutions for the logistic utility

In order to find the optimal consumption each agent i maximizes

$$U_i = -\frac{q^2}{2} + \sum_{j\neq i} x_{ij}\,(1-x_{ij})\; + \lambda_i\left(M_{i,t-1} - \sum_{j\neq i} p_j x_{ij}\right). \tag{A.15}$$

The calculation is the same as in Sec. 2.4.1 and gives the result

$$\lambda_i = \frac{1}{\sum_{k\neq i} p_k^2}\Big(\sum_{k\neq i} p_k - 2M_{i,t-1}\Big) \qquad \text{and} \qquad x_{ij} = \frac{1}{2}(1-\lambda_i p_j). \tag{A.16}$$

Putting the two results together:

$$x_{ij} = \frac{1}{2} - p_j\frac{\sum_{k\neq i} p_k}{2\sum_{k\neq i} p_k^2} + M_i p_j \frac{1}{\sum_{k\neq i} p_k^2}. \tag{A.17}$$

Although x_{ij} is not proportional to M_i, the homogenous solution gives

$$x = \frac{M/p}{N-1} \tag{A.18}$$

as it should and λ becomes

$$\lambda = \frac{1}{p} - 2\frac{M}{(N-1)p^2}. \tag{A.19}$$

As in Sec.A.1.1 one can assume that the reaction Eq. A.16 to given prices is known and calculate the optimal price by substituting q and $M_{i,t-1}$ in Eq. A.15. In the large N limit where the dependence of $\sum_{k\neq j} 1/p_k$ and $\sum_{k\neq j} 1/p_k^2$ on p_i is neglected the result is:

$$p_i = \frac{\frac{N-1}{2}\left(\lambda_i + \frac{1}{2}\sum_{k\neq i}\lambda_k\right)}{\frac{1}{4}\left(\sum_{k\neq i}\lambda_k\right)^2 + \lambda_i\sum_{k\neq i}\lambda_k}. \tag{A.20}$$

In the homogenous case:

$$p = \frac{(N+3)M}{N-1}. \tag{A.21}$$

With this result we see that x is simply given by $\frac{1}{N+3}$ which gives an equilibrium utility of

$$U = \left(\frac{1}{N+3}\right)^2 (1/2N^2 + 2N - 2.5) \tag{A.22}$$

which is a monotonically increasing function of N that converges to 0.5 in the large N limit, see Fig. A.2. Compare this result to the BOTTOM figure of Fig. 3.12.

A.3 Random walkers, an example of an interesting bug

This section is one more demonstration to show how easy it was to get things wrong when the small, at first sight nonproblematic change of the consumption utility (from square root to logistic) was made. In this case, the mistake leaded also to a very behaviour of the agents.

Again the problem is related to the fact that in case of logistic consumption if an agent j charges a high price, it is possible that $\lambda_i p_j > 1$ which means that x_{ij} is smaller than zero[1].

The most simple thing to do is setting those x_{ij} that would be negative to zero and rescale all x_{ij} in the end so that still all the money is spent in every timestep. Although this procedure seems to be reasonable at first sight, there is an important bug hidden in it.

First of all, the rescaling procedure is hard to justify, as it has to be done with a rescaling factor proportional to the amount of money which has to be spent. This would contradict Eq. 3.3 which states that the demands are not chosen proportionally.

Secondly, consider that all agents who charge too high prices and do not sell anything, will see their production, their amount of money and therefore also their utility drop to zero. These agents perform a random walk, accepting every price they test, because their utility does not change. In this way it is possible that their price becomes very large. This should of course not influence the behaviour of all the other agents, especially it should not influence the average price level, as these agents do not produce anything and the average price is given by

$$p_{av} = \frac{\sum q_i p_i}{\sum q_i}. \tag{A.23}$$

If q_i is zero, p_{av} does not depend on p_i. The simulation shows, though, that the average price level very much depends on the prices which these random walkers charge! In retrospect the reason for the strange behaviour is quite simple: Although not appearing explicitly in the formula for p_{av} the prices of the random walkers still appear in the calculation of the λ_i's and therefore influence all q_i. Only the iterative method described in Sec. 3.2.1 removes the bug and makes a correct calculation of the x_{ij} possible.

Now, strictly speaking, it makes no sense to let these random walkers exist at all: Every real "intelligent" agent that does not sell anything would lower her price as long as her production stays at zero. She would certainly not consider higher prices as well and in doing so become a random walker. But unless we allow our agents to be "stupid" random walkers, the bug will hardly become visible, as

[1] impossible in the basic model, see the corresponding Eq. 2.24.

no agents with unreasonably high prices will emerge! In fact, if non-producers are forced to lower their price (i.e. no random walkers can emerge), the simulation results with and without the iterative method will be almost identical, see Fig. A.3: The bug shows only if the agents are allowed to behave unreasonably.

The following figures will show the very interesting behaviour of this "bugged" economy and the analysis performed to get the bug uncovered. Fig. A.4 shows the curves for price, production, utility and money devlopment in an economy of 100 agents. Equilibrium and the (for logistic utility typical) fluctuations seem to start only around timestep 500000. Before that, strong fluctuations in the *average* price level are observed, accompanied by corresponding fluctuations in average production and utility. In this phase, individual behaviour shows essentially no deviation from the average! In the system of 500 agents, average prices can become as large as 150 before equilibrium starts, see Fig. A.5.

Now let's see how this strange behaviour is related to the presence of random walkers. Fig. A.6 shows that only as long as there are random walkers, the system shows the strange behaviour. As soon as there are no random walkers anymore, the system can come into equilibrium. As in this simulation there is essentially always only one random walker present (see Fig. A.6), it is easy to see the effect that these cause. For example Fig. A.7 show how the average price in the system always follows the price which the random walker charges! Indeed, if one agent a charges a very high price, x_{ij} is approximately given by

$$x_{ij} \approx \frac{1}{2} - \frac{p_j}{p_a} \frac{\sum_{k \neq i} p_k}{2p_a} \tag{A.24}$$

for $i, j \neq a$ (see below). Thus, although agent a does not have anything to do (directly) with the trades between agents i and j, x_{ij} depends strongly on p_a! With the above choice of their demands the utility of all agents is at a local maximum if prices are considered fixed for the moment. If agent a now happens to increase (decrease) her price, all x_{ij} do decrease (increase). In order to come back to the local maximum of utility, the agents must try to increase (decrease) their x_{ij}'s again. From the above formula one sees that this can be done by a collective increase (decrease) of the prices. In this way the average price level in the system always follows the random walk of agent a and no equilibrium can be established until all random walkers are "caught" by the system. As soon as all random walkers are caught, the distribution in prices gets very narrow (see Fig. A.8 and hence no agent can "escape" the system and become a random walker. Of course, the narrowness of the distribution of prices in equilibrium is also the reason for the large fluctuations that can be observed there, as, if the distribution is very narrow, agents that test prices are very "exposed". From now on the system behaves "normally", i.e. as the system in Sec 3.3.2, as the bug in the code has no effects anymore.

A.3.1 Demands in case of a random walker

These are approximations to Eq. 3.3 if there is a random walker "a" which charges a much too high price. If $i \neq a$, $\sum_{k \neq i} p_k^2$ is approximately given by p_a^2. In equilibrium the first two summands of Eq. A.17 can almost cancel to zero and only the third summand is important. In the present situation, though, M_i can safely be neglected compared to $\sum_{k \neq i} p_k$: if $i \neq a$ at least one summand is much larger than M_i, (M_i being of the order of the equilibrium price) and if $i = a$ M_i is zero anyway. We now get for the case of $i \neq a$ and $j \neq a$ that x_{ij} is approximately given by

$$x_{ij} \approx \frac{1}{2} - \frac{p_j}{p_a} \frac{\sum_{k \neq i} p_k}{2p_a}. \tag{A.25}$$

If j is the random walker itself we have

$$x_{ia} \approx -\frac{\sum_{k\neq i,j} p_k}{2p_a} + \frac{M_i}{p_a}. \tag{A.26}$$

Also here the second term is much smaller than the first, so x_{ia} is negative and will be set to zero. This means that no agent will buy from the random walker. Finally, if i is the random walker, x_{aj} will be set to zero in the rescale procedure of the program, as M_i is zero.

Bibliography

[1] P. Bak, S.F. Norrelykke, and M. Shubik. Dynamics of money. *Physical Review E*, 60(3):2528–2532, 1999.

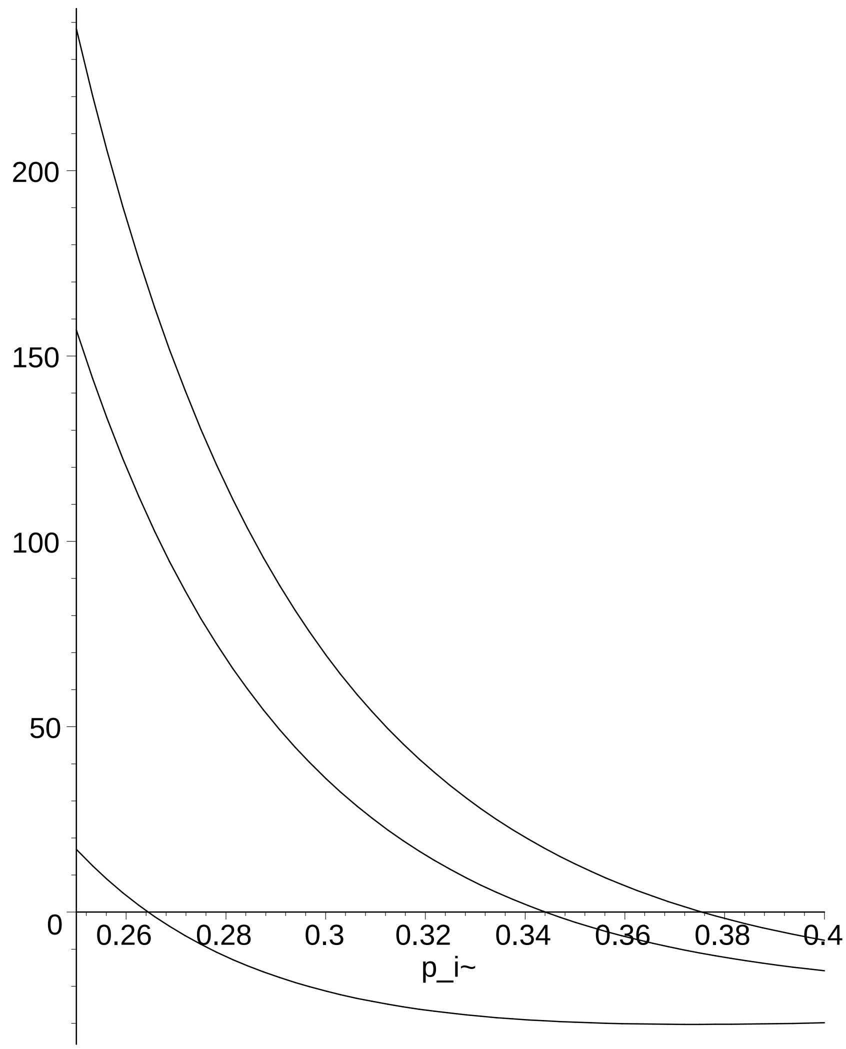

Figure A.1: *Optimal price for one agent in case all other agents charge price* p *is given by the root of the derivative of Eq. A.14. From left to right this derivative is plotted for* $N = 100$, $M_{tot} = N$ *and* $p = pSO \approx 0.216$, $p = p_{Nasheq} \approx 0.343$ *and* $p = 0.4$. *Note that the optimal price is always between* p_{Nasheq} *and* p.

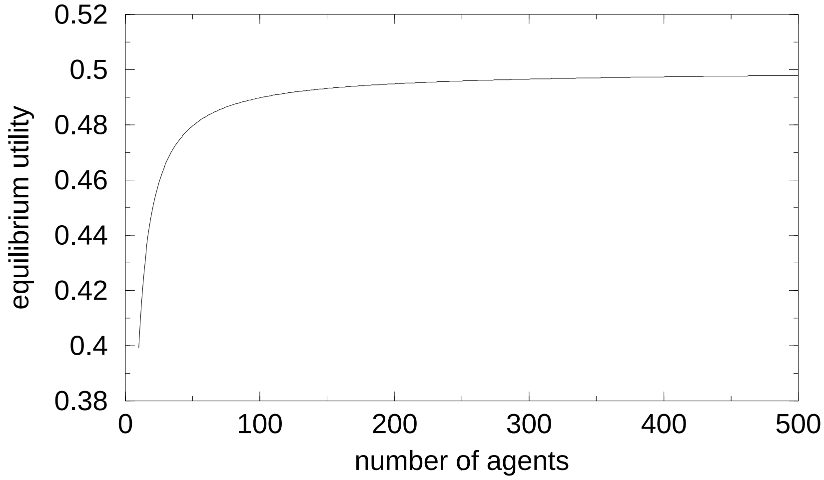

Figure A.2: *Analytically predicted value of the equilibrium utility in the large N limit as a function of N.*

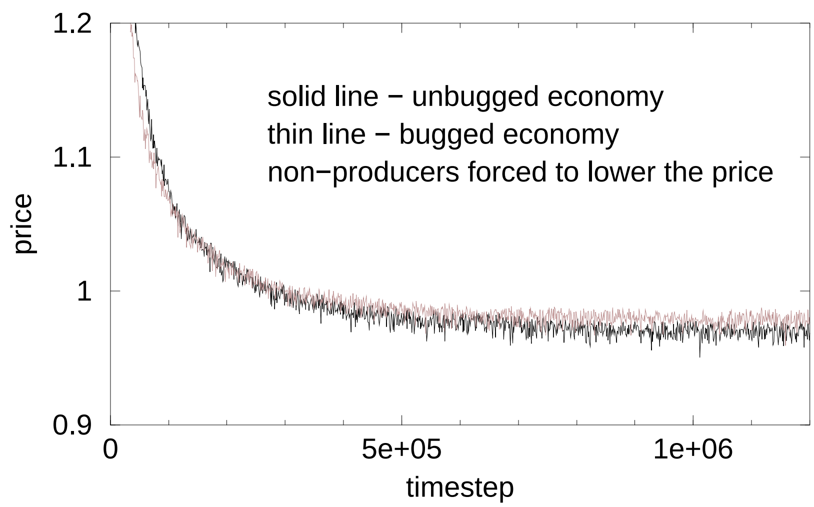

Figure A.3: *If non-producers are forced to lower their price (i.e. "intelligent agents"), the bug in the code is hidden as there are no random walkers: Thin curve is the average price in the bugged economy (i.e. here, if agent i buys nothing from agent j, p_j is not excluded from the calculation of λ_i). The solid curve is the average price in the simulation where the iterative method of Sec. 3.2.1 is used for the calculation of the λ's. Simulations with 100 agents.*

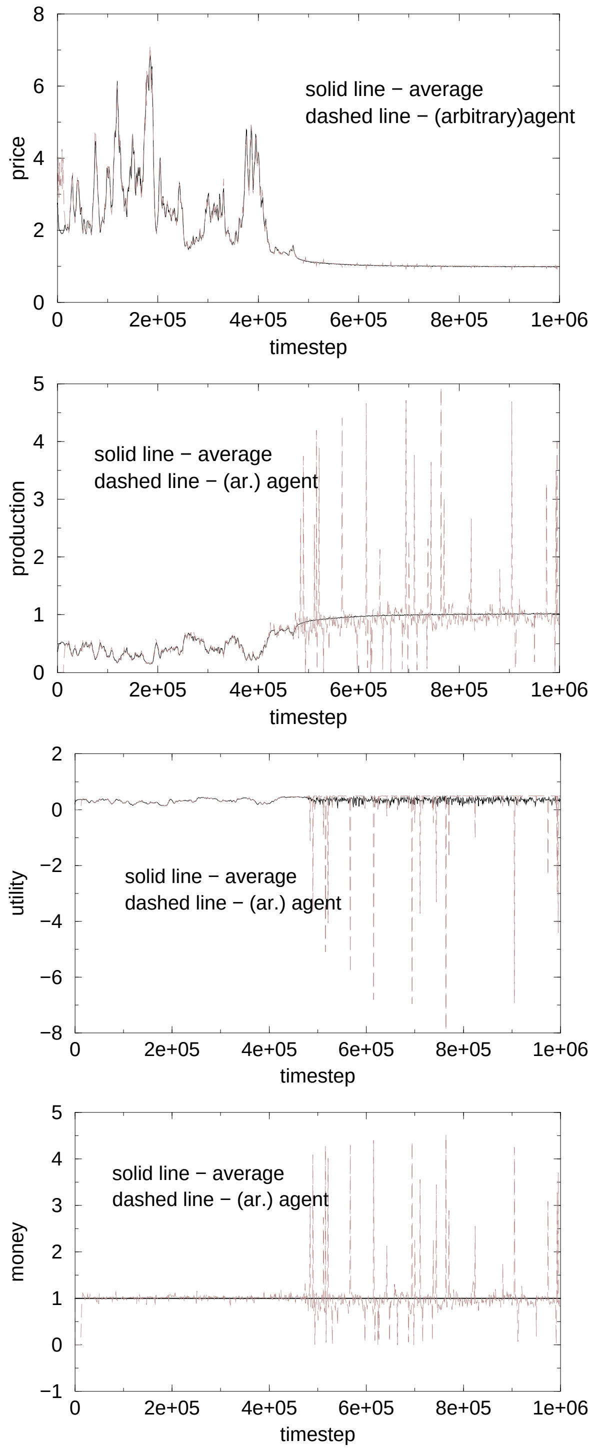

Figure A.4: *Relaxation for a logistic consumption utility when demands are calculated and non-producers are not forced to lower their price. System with 100 agents. Note the start of the fluctuations as soon as equilibrium is reached.*

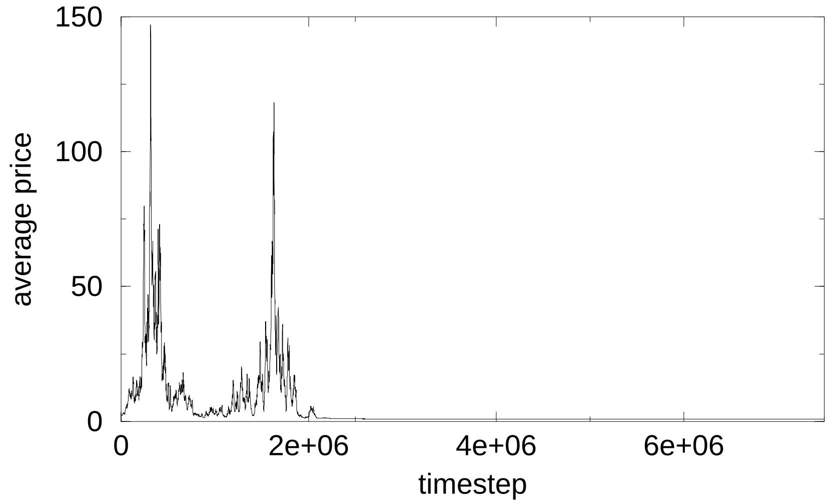

Figure A.5: *Relaxation for a logistic consumption utility when demands are calculated and non-producers are not forced to lower their price. System with 500 agents. Prices can be pushed as high as 150 by the presence of random walkers, but eventually all random walkers are caught.*

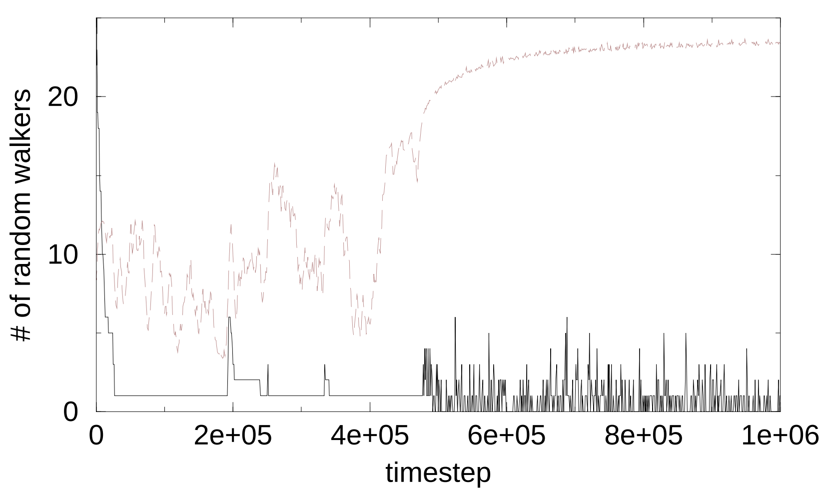

Figure A.6: *Number of agents which do not produce anything in the system of 100 agents. This is just the number of agents that charge a too high price and are random walkers therefore. The point where equilibrium is reached is marked by displaying the graph for average production, scaled up for better visibility. Before equilibrium is reached there are long periods with just one random walker: One agent is a random walker between timestep 0 to about 2.15e5 and there's a different one from about 1.95e5 until equilibrium is established, see Fig. A.7. Hence there are no random walkers anymore. The production of an agent can temporarily drop to zero, though, when she tests a higher price.*

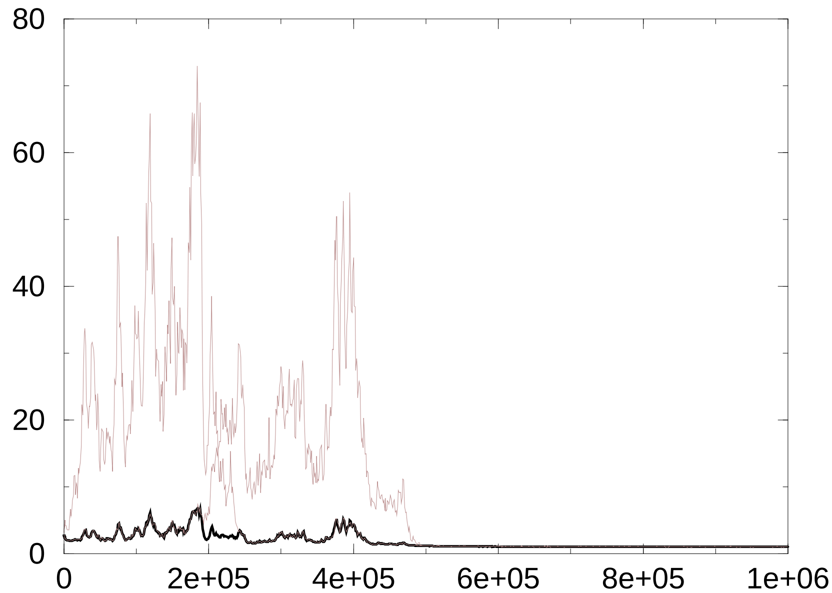

Figure A.7: *The figure shows the prices of the two random walkers mentioned in Fig. A.6 and their effect on the average price level.*

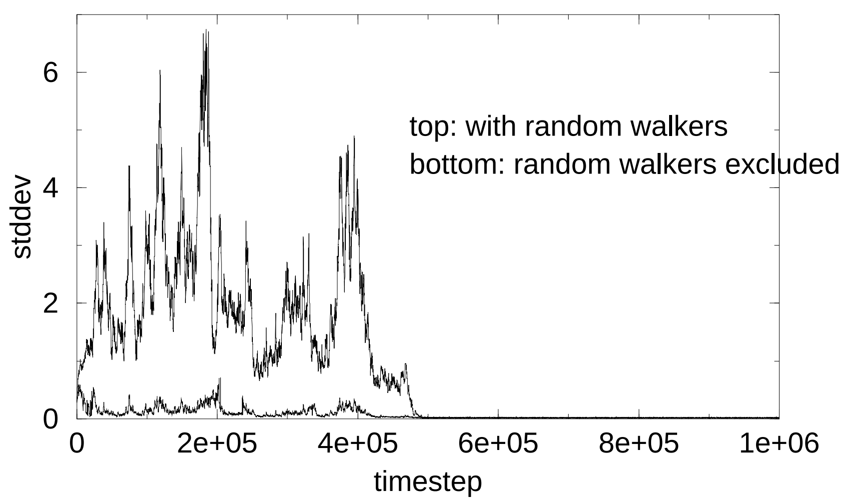

Figure A.8: *The figure shows the standard deviation in the distribution of prices as function of time. In the lower curve the random walkers are excluded from the calculation. Even without the random walkers, the distribution of prices cannot be narrow as long as there is no equilibrium, as the random walkers greatly disturb the price developement of all the other agents. Only when all the random walkers are caught and equilibrium is established, the standard deviation becomes very small.*