

Spatial competition and price formation

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Abstract

We look at price formation in a retail setting, that is, companies set prices, and consumers either accept prices or go someplace else. In contrast to most other models in this context, we use a two-dimensional spatial structure for information transmission, that is, consumers can only learn from nearest neighbors. Many aspects of this can be understood in terms of generalized evolutionary dynamics. In consequence, we first look at spatial competition and cluster formation without price. This leads to establishment size distributions, which we compare to reality. After some theoretical considerations, which at least heuristically explain our simulation results, we finally return to price formation, where we demonstrate that our simple model with nearly no organized planning or rationality on the part of any of the agents indeed leads to an economically plausible price.

Key words: econophysics; economics; simulation; markets

1 Introduction

There are several basic concepts which lie at the heart of economic theory. They are the "economic atom" which is usually considered to be the individual, profits, money, price and markets and the more complex organism the firm. Much of economic theory is based on utility maximizing individuals and profit maximizing firms. The concept of a utility function attributes to individuals a considerable amount of sophistication. The proof of its existence poses many difficult problems

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in observation and measurement. In this study of market and price formation we consider simplistic social individuals who must buy to eat and who look for where to shop for the best price. In this foray into dynamics we opt for a simple model of consumer price formation. Our firms are concerned with survival rather than a sophisticated profit maximization. Yet we relate these simple behaviors to the more conventional and complex ones.

A natural way to approach the economic physics of monopolistic competition is to introduce space explicitly. For much of economic analysis of competition space and information are critical factors. The basic aspects of markets involve an intermix of factors, such as transportation costs and delivery times which depend explicitly on physical space. But for pure information, physical distance is less important than direct connection. For questions concerning the growth of market areas, the spatial representation is appropriate. Consideration of space is sufficient to provide a justification of Chamberlin's model of monopolistic competition as is evident from the work of Hotelling [1]. Furthermore it is reasonably natural to consider space on a grid with some form of minimal distance. Many of the instabilities found in economic models such as the Bertrand model are not present with an appropriate grid.

When investigating these topics, one quickly finds that many aspects of price formation can be understood in terms of generalized evolutionary dynamics. In consequence, our first models in this paper study spatial competition and cluster formation without the generation of price (Sec. 3). This generates cluster size distributions, which can be compared to real world data. We spend some time investigating theoretical models which can explain our simulation data (Sec. 4). We then, finally, move on to price formation, where we implement the price dynamics "on top" of the already analyzed spatial competition models (Sec. 5). The paper is concluded by a discussion and a summary.

2 Related work

The model is an open one related to the partial equilibrium models of much of micro-economics. In particular money and its acceptance in trade is taken as a primitive concept. There is a literature on the acceptance of money both in a static equilibrium context (see for example [2]) and in a "bootstrap" or dynamic context (see for example [3,4]). These are extremely simple closed models of the economy where each individual is both a buyer and seller. Eventually we would like to construct a reasonable model where the acceptance of money, the emergence of competitive price and the emergence of market structure all arise from the system dynamics. This will call for an appropriate combination of the features of the model presented here with the closed models noted above. We do not pursue this further here. Instead by taking the acceptance of money as given our observations are con-

finer to the emergence of markets and the nature of price. The static economic theories of monopoly and mass homogeneous competitive equilibrium provide natural upper and lower benchmarks to gauge market behavior. The intermediate zone between $n = 2$ and very many is covered in the economic literature by various oligopoly models, of which those of Cournot [5], Bertrand [6] and Chamberlin [7] serve as exemplars. The Chamberlin model unlike the earlier models stresses that all firms trade in differentiated goods. They are all in part differentiated or partially monopolistic. When one considers both information and physical location this is a considerable step towards greater realism. Other work on evolutionary or behavioral learning in price formation are Refs. [8–10].

3 Spatial competition

As mentioned in the introduction, we will start with spatial models without price. We will add price dynamics later.

3.1 Basic spatial model (domain coarsening)

We use a 2-dimensional $N = L \times L$ grid with periodic boundary conditions. Sites are numbered $i = 1..N$. Each site belongs to a cluster, denoted by $c(i)$. Initially, each site belongs to “itself”, that is, $c(i) = i$, and thus cluster numbers also go from 1 to N .

The dynamics is such that in each time step we randomly pick a cluster, delete it, and the corresponding sites are taken over by neighboring clusters. Since the details, in particular with respect to the time scaling, make a difference, we give a more technical version of the model. In each time step, we first select a cluster for deletion by randomly picking a number C between 1 and N . All sites belonging to the cluster (i.e. $c(i) = C$) are marked as “dead”. We then let adjoining clusters grow into the “dead” area. Because of the interpretation later in the paper, in our model the “dead” sites play the active role. In parallel, they all pick randomly one of their four nearest neighbors. If that neighbor is not dead (i.e. belongs to a cluster), then the previously dead site will join that cluster. This step is repeated over and over, until no dead sites are left. Only then, time is advanced and the next cluster is selected for deletion.

In physics this is called a domain coarsening scheme (e.g. [11]): Clusters are selected and deleted, and their area is taken over by the neighbors. This happens with a total separation of time scales, that is, we do not pick another cluster for deletion before the distribution of the last deleted cluster has finished. Fig. 1 shows an example. We will call a cluster of size larger than zero “active”.

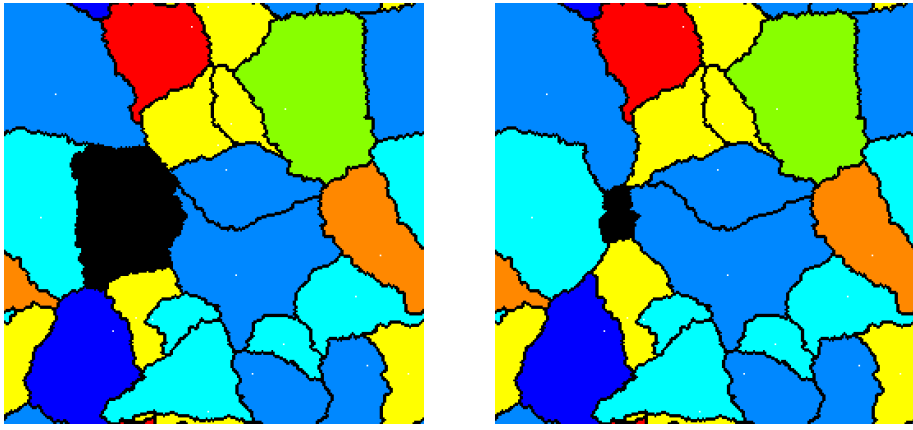


Fig. 1. Snapshot of basic domain coarsening process. LEFT: The black space comes from a cluster that has just been deleted. RIGHT: The black space is being taken over by the neighbors. — Colors/grayscale are used to help the eye; clusters which have the same color/grayscale are still different clusters. System size 256^2 .

Note that it is possible to pick a cluster that has already been deleted. In that case, nothing happens except that the clock advances by one. This implies that there are two reasonable definitions of time:

- **Natural time t :** This is the definition that we have used above. In each time step, the probability of any given cluster to be picked for deletion is a constant $1/N$, where $N = L^2$ is the system size. Note that it is possible to pick a cluster of size zero, which means that nothing happens except that time advances by one.
- **Cluster time \tilde{t} :** An alternative is to choose between the *active* clusters only. Then, in each time step, the probability of any given cluster to be picked for deletion is $1/n(\tilde{t})$, where $n(\tilde{t}) = N - \tilde{t}$ is the number of remaining active clusters in the system.

Although the dynamics can be described more naturally in cluster time, we prefer natural time because it is closer to our economics interpretation.

At any particular time step, there is a typical cluster size. In fact, in cluster time, since there are $n(\tilde{t}) = N - \tilde{t}$ clusters, the average cluster size as a function of cluster time is $\bar{S}(\tilde{t}) = N/n(\tilde{t}) = 1/(1 - \tilde{t}/N)$. However, if one *averages over all time steps*, we find a scaling law. In cluster time, it is numerically close to $\tilde{n}(s) \sim s^{-3}$ or $\tilde{n}(s) \sim s^{-2}$, where s is the cluster size, $n(s)$ is the number of clusters of size s , and $n(>s)$ is the number of clusters with size larger than s .⁵ In natural time, the large clusters have more weight since time moves more slowly near the end of the coarsening process. The result is again a scaling law (Fig. 2 (left)),

⁵ In this paper, we will also use $N(s) = s n(s)$ for the cluster size distribution in logarithmic bins, in particular for the figures.

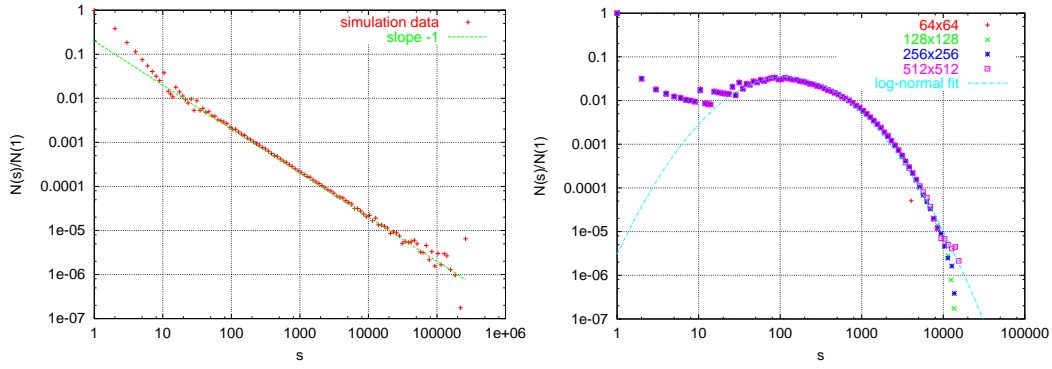


Fig. 2. LEFT: Cluster size distribution of the basic model without injection, in natural time. Number of clusters per logarithmic bin, divided by number of clusters in first bin. The straight line has slope -1 , corresponding to $n(s) \sim s^{-2}$ because of logarithmic bins. System size 512^2 . As explained in the text, this is not a steady state distribution, but a distribution which emerges when averaging over the complete evolution from N clusters of size one to one cluster of size N . RIGHT: Cluster size distribution for random injection. Number of clusters per logarithmic bin, divided by number of clusters in first bin. The plot shows $p_{inj} = 0.01$ and system sizes 64^2 , 128^2 , 256^2 , and 512^2 . The line is a log-normal fit. This is a steady state distribution.

but with exponents increased by one:

$$n(s) \sim s^{-2} \text{ or } n(>s) \sim s^{-1}. \quad (1)$$

It is important to note that this is not a steady state result. The result emerges when averaging over the whole time evolution, starting with N clusters of size one and ending with one cluster of size N .

3.2 Random injection with space

In view of evolution, for example in economics or in biology, it is realistic to inject new small clusters. A possibility is to inject them at random positions. So in each time step, before the cluster deletion described above, in addition with probability p_{inj} we pick one random site i and inject a cluster of size one at i . That is, we set $c(i)$ to i . This is followed by the usual cluster deletion. It will be explained in more detail below what this means in terms of system-wide injection and deletion rates.

This algorithm maintains the total separation of time scales between the cluster deletion (slow time scale) and cluster growth (fast time scale). That is, no other cluster will be deleted as long as there are still “dead” sites in the system. Note that the definition of time in this section corresponds to natural time.

The probability that the injected cluster is really new is reduced by the probabil-

ity to select a cluster that is already active. The probability of selecting an already active cluster is $n(t)/N$, where $n(t)$ is again the number of active clusters. In consequence, the effective injection rate is

$$r_{inj,eff} = p_{inj} - n(t)/N . \quad (2)$$

Similarly, the effective cluster deletion depends on the probability of picking an active cluster, which is $n(t)/N$. In consequence, the effective deletion rate is

$$r_{del,eff} = n(t)/N . \quad (3)$$

This means that, in the steady state, there is a balance of injection and deletion, $n_*/N = p_{inj} - n_*/N$, and thus the steady state average cluster number is

$$n_* = N p_{inj} / 2 . \quad (4)$$

In consequence, the steady state average cluster size is

$$s_* = N/n_* = 2/p_{inj} . \quad (5)$$

The cluster size distribution for the model of this section is numerically close to a log-normal distribution, see Fig. 2 (right). Indeed, the position of the distribution moves with $1/p_{inj}$ (not shown). In contrast to Sec. 3.1, this is now a steady state result.

3.3 Injection on a line

It is maybe intuitively clear that the injection mechanism of the model described in Sec. 3.2 destroys the scaling law from the basic model without injection (Sec. 3.1), since injection at random positions introduces a typical spatial scale. One injection process that actually generates steady-state scaling is injection along a 1-d line. Instead of the random injection of Sec. 3.2, we now permanently set

$$c(i) = i \quad (6)$$

for all sites along a line. Fig. 3 (left) shows a snapshot of this situation.

In this case, we numerically find a stationary cluster size distribution (Fig. 3 (right)) with

$$n(s) \sim s^{-1.5} \text{ or } n(>s) \sim s^{-0.5} . \quad (7)$$

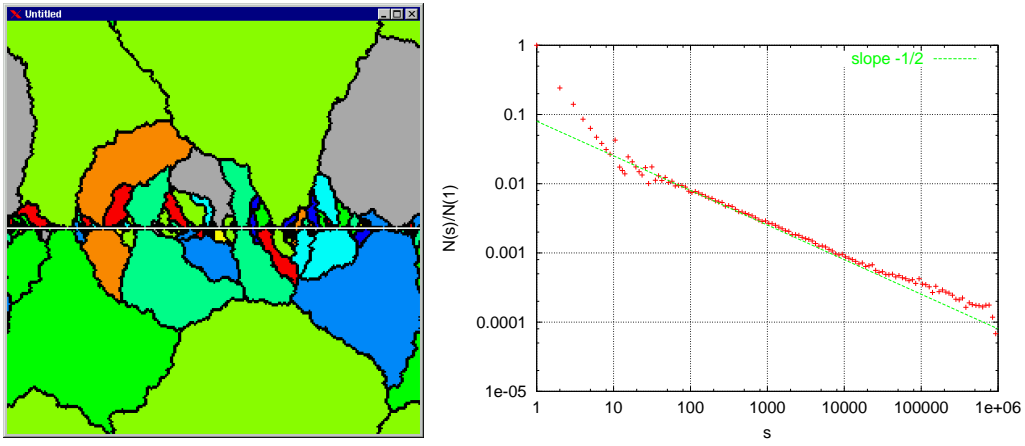


Fig. 3. LEFT: Injection along a line. System size 256^2 . RIGHT: Scaling plot for basic model plus injection on a line. Number of clusters per logarithmic bin, divided by number of clusters in first bin. The straight line has slope $-1/2$ corresponding to $n(s) \sim s^{-3/2}$. System size 1024^2 . This is a steady state distribution.

Since the injection mechanism here does not depend on time, and since the cluster size distribution itself is stationary, it is independent from the specific definition of time.

3.4 Random injection without space

One could ask what would happen without space. A possible translation of our model into “no space” is: Do in parallel: Instead of picking one of your four nearest neighbors, you pick an arbitrary other agent (random neighbor approximation). If that agent is not dead, copy its cluster number. Do this over and over again in parallel, until all agents are part of a cluster again. A cluster is now no longer a spatially connected structure, but just a set of agents. In that case, we obtain again power laws for the size distribution, but this time with slopes that depend on the injection rate p_{inj} (Fig. 4); see Sec. 4.4 for details.

3.5 Real world company size distributions

Fig. 5 shows actual retail company size distributions from the 1992 U.S. economic census [12], using annual sales as a proxy for company size. We use the retail sector because we think that it is closest to our modelling assumptions — this is discussed at the end of Sec. 6. We show two curves: establishment size, and firm size.⁶ It is

⁶ An establishment is “a single physical location at which business is conducted. It is not necessarily identical with a company or enterprise, which may consist of one establishment or more.” [12].

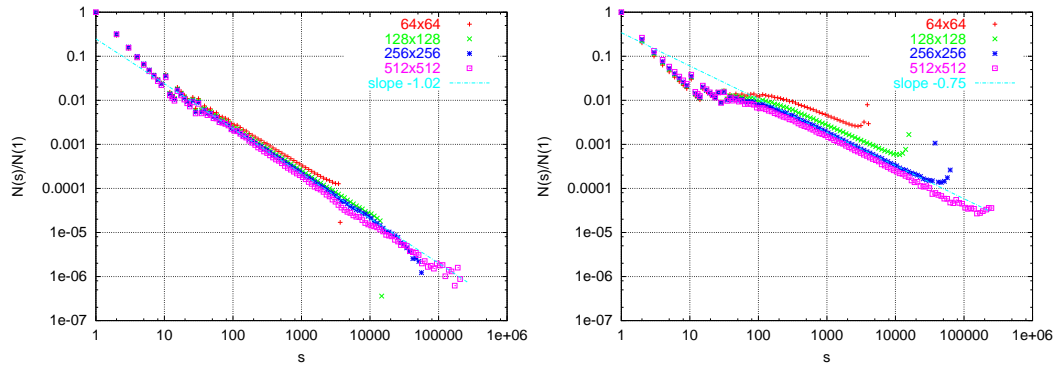


Fig. 4. Steady state cluster size distributions for different non-spatial simulations. Number of clusters per logarithmic bin, divided by number of clusters in first bin. System sizes 64^2 to 512^2 . LEFT: $p_{inj} = 0.1$. RIGHT: $p_{inj} = 0.01$.

clear that in order to be comparable with our model assumptions, we need to look at establishment size rather than at company size.

Census data comes in unequally spaced bins; the procedure to convert it into useable data is described in the appendix. Also, the last four data points for firm size (not for the establishment size, however) were obtained via a different method than the other data points; for details, again see the appendix.

From both plots, one can see that there is a typical establishment size around \$400 000 annual sales; and the typical firm size is a similar number. This number intuitively makes sense: With, say, income of 10% of sales, smaller establishments will not provide a reasonable income.

One can also see from the plots that the region around that typical size can be fitted by a log-normal. We also see, however, that for larger numbers of annual sales, such a fit is impossible since the tail is much fatter. A scaling law with

$$n(>s) \sim s^{-1} \text{ corresponding to } n(s) \sim s^{-2} \quad (8)$$

is an alternative here.⁷

This is, however, at odds with investigations in the literature. For example, Ref. [13] find a log-normal, and by using a Zipf plot they show that for large companies the tail is *less* fat than a log-normal. However, there is a huge difference between our and their data: They only use *publicly traded* companies, while our data refers to all companies in the census. Indeed, one finds that their plot has its maximum at annual sales of 10^8 , which is already in the tail of our distribution. This implies

⁷ Remember again, that slopes from log-log plots in logarithmic bins are different by one from the exponent in the distribution. So $n(s) \sim s^{-2}$ corresponds to a slope -1 both in the accumulated distribution $n(>s)$ and when plotting logarithmic bins $N(s)/N(1)$.

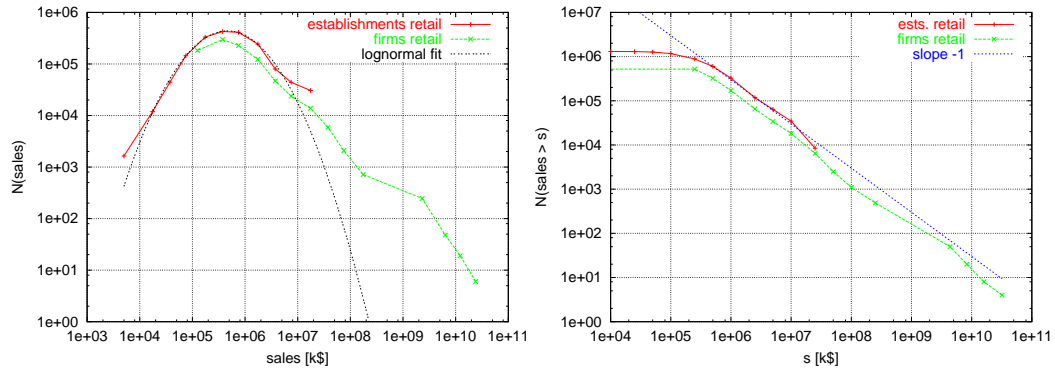


Fig. 5. 1992 U.S. Economic Census data. LEFT: Number of retail establishments/retail firms per logarithmic bin as function of annual sales. RIGHT: Number of establishments/firms which have more sales than a certain number.

that the small scale part of their distribution comes from the fact that small companies are typically not publicly traded. In consequence, it reflects the dynamics of companies entering and exiting from the stock market, not entry and exit of the company itself.

We conclude that from available data, company size distributions are between a log-normal and a power law with $n(s) \sim s^{-2}$ or $n(>s) \sim s^{-1}$. Further investigation of this goes beyond the scope of this paper.

4 Theoretical considerations

4.1 Spatial coarsening model (slope -2 in natural time)

We are looking again at the “basic model”. In cluster time this was: randomly pick one of the clusters, and give it to the neighbors. The following heuristic model gives insight:

- (1) We start with N clusters of size 1.
- (2) We need $N/2$ time steps to delete $N/2$ of them and with that generate $N/2$ clusters of size 2.
- (3) In general, we need $N/2^k$ time steps to move from $N/2^{k-1}$ clusters of size 2^{k-1} to $N/2^k$ clusters of size 2^k .
- (4) If we sum this over time, then in each logarithmic bin at $s = 2^k$ the number of contributions is $N/2^k \times N/2^k$, i.e. $\sim s^{-2}$.
- (5) Since these are logarithmic bins, this corresponds to $\tilde{n}(s) \sim s^{-3}$ or $\tilde{n}(>s) \sim s^{-2}$, which was indeed the simulation result in cluster time.
- (6) In natural time, we need a constant amount of time to move from $k-1$ to k , and thus obtain via the same argument $n(s) \sim s^{-2}$ or $n(>s) \sim s^{-1}$, which

was the simulation result in natural time.

4.2 Random injection in space (log-normal)

At the moment, we do not have a consistent explanation for the log-normal distribution in the spatial model. A candidate is the following: Initially, most injected clusters of size one are *within* the area of some larger and older cluster. Eventually, that surrounding cluster gets deleted, and all the clusters of size one spread in order to occupy the now empty space. During this phase of fast growth, the speed of growth is proportional to the perimeter, and thus to \sqrt{s} , where s is the area. Therefore, \sqrt{s} follows a biased multiplicative random walk, which means that $\log(\sqrt{s}) = \log(s)/2$ follows a biased additive random walk. In consequence, once that fast growth process stops, $\log(s)$ should be normally distributed, resulting in a log-normal distribution for s itself. In order for this to work, one needs that this growth stops at approximately the same time for all involved clusters. This is approximately true because of the “typical” distance between injection sites which is inversely proportional to the injection rate. More work will be necessary to test or reject this hypothesis.

4.3 Injection on a line (slope -3/2)

If one looks at a snapshot of the 2D picture for “injection on a line” (Fig. 3), one recognizes that one can describe this as a structure of cracks which are all anchored at the injection line. There are L such cracks (some of length zero); cracks merge with increasing distance from the injection line, but they do not branch.

According to Ref. [14], this leads naturally to a size exponent of $-3/2$, as found in the simulations. The argument is the following: The whole area, L^2 , is covered by

$$\int ds s n(s) , \quad (9)$$

where $n(s)$ is the number of clusters of size s on a linear scale. We assume $n(s) \sim s^{-\tau}$, however the normalization is missing. If all clusters are anchored at a line of size L , then a doubling of the length of the line will result in twice as many clusters. In consequence, the normalization constant is $\propto L$, and thus $n(s) \sim L s^{-\tau}$. Now we balance the total area, L^2 , with what we just learned about the covering clusters:

$$L^2 \sim \int ds s L s^{-\tau} = L \int ds s^{1-\tau} \sim L s^{2-\tau} \Big|_0^S . \quad (10)$$

Assuming that $\tau < 2$, then the integral does not converge for $S \rightarrow \infty$, and we

need to take into account how the cut-off S scales with L . This depends on how the cracks move in space as a function of the distance from the injection line. If the cracks are roughly straight, then the size of the largest cluster is $\sim L^2$. If the cracks are random walks, then the size of the largest cluster is $\sim L^{3/2}$. In consequence:

- For “straight” lines: $L^2 \sim L (L^2)^{2-\tau} \Rightarrow 2 = 1 + 2(2 - \tau) \Rightarrow \tau = 3/2$.
- For random walk: $2 = 1 + 3/2(2 - \tau) \Rightarrow \tau = 4/3$.

Since our simulations result in $\tau \approx 3/2$, we conclude that our lines between clusters are not random walks. This is intuitively reasonable: When a cluster is killed, then the growth is biased towards the center of the deleted cluster, thus resulting in random walks which are all differently biased. This bias then leads to the “straight line” behavior. — This implies that the $\sim s^{-3/2}$ steady state scaling law hinges on two ingredients (in a 2D system): (i) The injection comes from a 1D structure. (ii) The boundaries between clusters follow something that corresponds to straight lines. As we have seen, the biasing of a random walk is already enough to obtain this effect.

4.4 Injection without space (variable slope)

Without space, clusters do not grow via neighbors, but via random selection of one of their members. That is, we pick a cluster, remove it from the system, and then give its members to the other clusters one by one. The probability that the agent choses a cluster i is proportional to that cluster’s size s_i . If for the moment we assume that time advances with each member which is given back, we obtain the rate equation

$$\frac{dn(s)}{dt} = (s-1)n(s-1) - sn(s) - \epsilon n(s) - \epsilon p_{inj} n(s) + \epsilon p_{inj} n(s+1). \quad (11)$$

The first and second term on the RHS represent cluster growth by addition of another member; the third term represents random deletion; the fourth and fifth term the decrease by one which happens if one of the members is converted to a start-up via injection. ϵ is the rate of cluster deletion; since we first give all members of a deleted cluster back to the population before we delete the next cluster, it is proportional to the inverse of the average cluster size and thus to the injection rate: $\epsilon \sim 1/\langle s \rangle \sim p_{inj}$. This is similar to an urn process with additional deletion.

Via the typical approximations $sn(s) - (s-1)n(s-1) \approx \frac{d}{ds}(sn(s))$ etc. we obtain, for the steady state, the differential equation

$$0 = -n - s \frac{dn}{ds} - \epsilon n + \epsilon p_{inj} \frac{dn}{ds}. \quad (12)$$

This leads to

$$n(s) \propto (s - \epsilon p_{inj})^{-(1+\epsilon)} \sim s^{-(1+\epsilon)}. \quad (13)$$

That is, the exponent depends on the injection rate, and in the limit of $p_{inj} \rightarrow 0$ it goes to -1 . This is indeed the result from Sec. 3.4 (see Fig. 4).⁸

5 Price formation

What we will do now is to add the mechanism of price formation to our spatial competition model. For this, we identify sites with consumers/customers. Clusters correspond to domains of consumers who go to the same shop/company. Intuitively, it is clear how this should work: Companies which are not competitive will go out of business, and their customers will be taken over by the remaining companies. The reduction in the number of companies is balanced by the injection of start-ups. Companies can go out of business for two reasons: losing too much money, or losing too many customers. The first corresponds to a price which is too low; the second corresponds to a price which is too high.

We model these aspects as follows: We again have N sites on an $N = L \times L$ grid with periodic boundary conditions (torus). On each site, we have a consumer and a firm. These are not connected in any way except by the spatial position – one can imagine that the firm is located “downstairs” while the consumer lives “upstairs”. Firms with customers are called “active”, the other ones “inactive”. A time step consists of the following sub-steps:

- Trades are executed.
- Companies with negative profit go out of business.
- Companies change prices.
- New companies are injected.
- Consumers can change where they shop.

These steps are described in more detail in the following:

Trade: All customers have an initial amount M of money, which is completely spent in each time step and replenished in the next. Every customer i also knows which firm $j = f(i)$ he/she buys from. Thus, he/she orders an amount $Q_i = M/P_j$

⁸ Note that the approach in this section corresponds to measuring the cluster size distribution every time we give an agent back to the system, while in the simulations we measured the cluster size distribution only just before a cluster was picked for deletion. In how far this is important is an open question; preliminary simulation results indicate that it is important for the spatial case with injection but not important for the non-spatial case in this section.

at his/her company, where P_j is that company's price. The companies produce to order, and then trades are executed. That is, a company that has n_j customers and price P_j will produce and sell $Q_j = n_j M / P_j$ units and will collect $n_j M$ units of money.

Company exit: We assume an externally given cost function for production, $C(Q)$, which is the same for everybody. If profit $\Pi_j := n_j M - C(n_j M / P_j)$ is less than zero, then the company is losing money and will immediately go out of business.⁹ The prices of such a company is set to infinity. We will use $C(Q) = Q$, corresponding to a linear cost of production. With this choice, companies with prices $P_j < 1$ will exit according to this rule as soon as they attract at least one customer.

Price changes: With probability one, pick a random integer number between 1 and N . If there is an active company with that number, its price is randomly increased or decreased by δ .

Company injection: Companies are made active by giving them one customer: With probability p_{inj} , pick a random site i and make the consumer i go shopping at company i . The price of the injected company is set to the price that the customer has paid before, randomly increased or decreased by δ .

Consumer adaptation: All customers whose prices got increased (either via “company exit” or via “price changes”) will search for a new shop.¹⁰ These “searching” consumers correspond to dead sites in the basic spatial models (Sec. 3), and the dynamics is essentially a translation of that: All searching consumers in parallel pick a random nearest neighbor. If that neighbor is also searching, nothing happens. If that neighbor is however not searching, and if that neighbor is paying a lower price, our consumer will accept the neighbor's shop. Otherwise the customer will remain with her old shop, and she will no longer search. We keep repeating this until no consumer is searching any more.

This model does not invest much in terms of rational or organized behavior by any of the entities. Firms change prices randomly; and they exit without warning when they lose money. New companies are injected as small variations of existing companies. Consumers only make moves when they cannot avoid it (i.e. their company went out of business and they need a new place to go shopping) or when prices just went up. Only in the last case they actively compare some prices. It will turn out (see below) that even that price comparison is not necessary.

In the above model, price converges to the unit cost of production, which is the competitive price. In Fig. 6 (left, bottom curve) we show how an initially higher

⁹ In this model no accumulation of assets is allowed. This simplification will be relaxed in future work.

¹⁰ The simplification that customers react to price changes only is useful because it leads to the separation of time scales between consumer behavior and firm behavior.

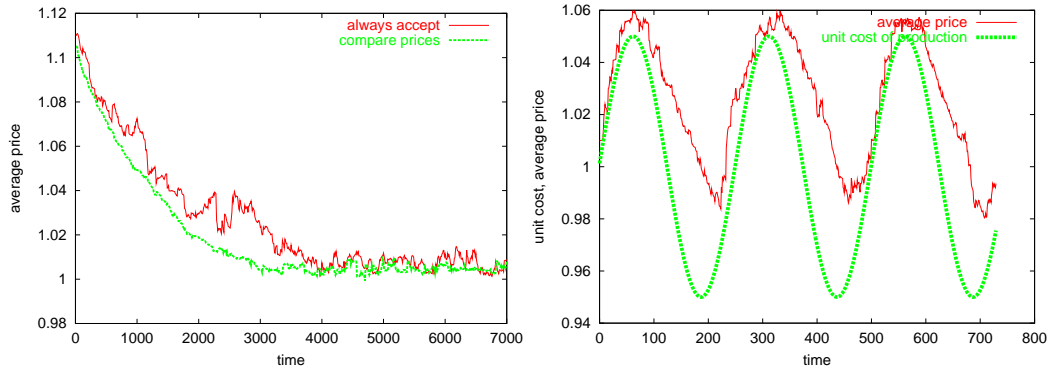


Fig. 6. LEFT: Price adjustment. Bottom curve: when searching consumers compare prices. Top curve: when searching consumers accept prices no matter what they are. RIGHT: Prices tracking the cost of production.

price slowly decreases towards a price of one. The reason for this is that, as long as prices are larger than one, there will be companies that, via random changes or injection, have a lower price than their neighbors. Eventually, these neighbors raise prices, thus driving their customers away and to the companies with lower prices. If, however, a company lowers its price below one, then it will immediately exit after it has attracted at least one customer.¹¹

As already mentioned above, it turns out that the price comparison by the consumers is not needed at all. We can replace the rule “if price goes up, try to find a better price” by “if price goes up, go to a different shop no matter what the price there”. In both cases, we find the alternative shop via our neighbors, as we have done throughout this paper. The top curve in Fig. 6 shows the resulting price adjustment. Clearly, the price still moves towards the critical value of one, but it moves more slowly and the trajectory displays more fluctuations. This is what one would expect, and we think it is typical for the situation: If we reduce the amount of “rationality”, we get slower convergence and larger fluctuations.

In terms of cluster size distribution, the price model is similar to the earlier spatial competition model with random injection. They would become the same if we separated bankruptcy and price changes.

In Fig. 6 (right) we also show that our model is able to track slowly varying costs of production. For this, we replace $C(Q) = Q$ by a sinus-function which oscillates around Q . The plot implies that prices lag behind the costs of production.

¹¹ If *all* prices in the system are more than δ below one, then the model is not well-defined. In the limit of large systems and when starting with prices above one, such a state cannot be reached via the dynamics. – Also note that if the model allowed credit, the exit of such a company would be delayed, allowing losses for limited periods of time.

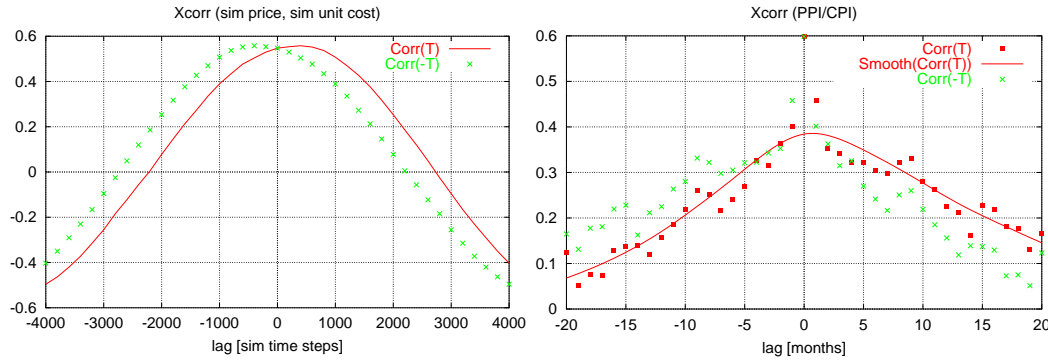


Fig. 7. Crosscorrelation function between R_{price} and R_{cost} : $R_x := x(t)/x(t-1)$; $Xcorr(\tau) := \langle R_P(t)R_C(t-\tau) \rangle$. LEFT: Simulation. The crosses show the crosscorrelation values mirrored at the $\tau = 0$ axis, in order to stress the asymmetry. RIGHT: U.S. Consumer price index for price and Producer price index for cost. Filled boxes are the crosscorrelation values; the smooth line is an interpolating spline for the filled boxes. The crosses show the crosscorrelation values mirrored at the $\tau = 0$ axis.

This is also visible in the asymmetry of the cross correlation function between both series. In order to be able to compare with non-stationary real world series, we look at relative changes, $R_x(t) = x(t)/x(t-1)$. The cross correlation function between price increases and cost increases then is

$$Xcorr(\tau) = \langle R_P(t) R_C(t - \tau) \rangle, \quad (14)$$

where $\langle . \rangle$ averages over all t . In Fig. 7 (left) one can clearly see that prices are indeed lagging behind costs for our simulations. In order to stress the asymmetry, we also plot $XCorr(-\tau)$. In Fig. 7 (right) we show the same analysis for the Consumer Price Index vs. the Production Price Index (seasonally adjusted). Although the data is much more noisy, it is also clearly asymmetric.

6 Discussion and outlook

The modelling approach with respect to economics in this paper is admittedly simplistic. Some obvious and necessary improvements concern credit and bankruptcy (i.e. rules for companies to operate with a negative amount of cash). Instead of those, we want to discuss some issues here that are closer to this paper. These issues are concerned with time, space, and communication.

In this paper, in order to reach a clean model with possible analytic solutions, we have described the models in a language which is rather unnatural with respect to economics. For example, instead of “one company per time step” which changes prices one would use rates (for example a probability of p_{ch} for each company to

change prices in a given time step). However, in the limiting case of $p_{ch} \rightarrow 0$, at most one and usually zero companies change prices in a given time step. If one also assumes that consumers adaptation is fast enough so that it is always completed before the next price change occurs, then this will result in the same dynamics as our model. Thus, our model is not “different” from reality, but it is a limiting case for the limit of fast customer adaptation and slow company adaptation. Our approach is to understand these limiting cases first before we move to the more general cases.

Similar comments refer to the utilization of space. We have already seen that moving from a spatial to a non-spatial model is rather straightforward. There is an even more systematic way to make this transition, which is the increase of the dimensions. In two dimensions on a square grid, every agent has four nearest neighbors. In three dimensions, there are six nearest neighbors. In general, if D is the dimension, there are $2D$ nearest neighbors. If we leave the number N of agents constant and keep periodic boundary conditions (D -dimensional torus), then at $D = (N - 1)/2$ everybody is a nearest neighbor of everybody. Thus, a non-spatial model is the $D \rightarrow \infty$ limiting case of a spatial model.¹²

These concepts can be generalized beyond grids and nearest neighbors – the only two ingredients one needs is that (i) the probability to interact with someone else decreases fast enough with distance, and that (ii) if one doubles distance from r to $2r$, then the number of interactions up to $2r$ is 2^D times the number of interactions up to distance r .

This should also make clear that space can be seen in a generalized way if one replaces distance by generalized cost. For example, how many more people can you call for “20 cents a minute or less” than for “10 cents a minute or less”? If the answer to this is “two times as many”, then for the purposes of this discussion you live in a one-dimensional world.

Given this, it is important to note that we have used space only for the communication structure, i.e. the way consumers acquire information (by asking neighbors). This is a rather weak influence of space, as opposed to, for example, transportation costs[1]; it however also assumes a not very sophisticated information structure, as for example in contrast to today’s internet. The details of this need to be left to future work.

Last, one needs to consider which part of the economy one wants to model. For example, a stockmarket is a centralized institution, and space plays a weak role at best. In contrast, we had the retail market in mind when we developed the models of this paper. In fact, we implicitly assume perishable goods, since agents have no

¹² Furthermore, models such as the ones discussed in this paper often have a so-called upper critical dimension, where some aspects of the model become the same as in infinite dimensions. This upper critical dimension often is rather low (below 10).

memory of what they bought and consumed the day before. Also, we assume that consumers spend little effort in selecting the “right” place to shop, which excludes major personal investments such as cars or furniture. Also note that our companies have no fixed costs, which implies that there are no capital investments, which excludes for example most manufacturing.

7 Summary

Price formation is an important aspect of economic activity. Our interest was in price formation in “everyday” situations, such as for retail prices. For this, we assumed that companies are price setters and agents are price takers, in the sense that their only strategy option is to go someplace else. In our abstracted situation, this means that companies with too low prices will exit because they cannot cover costs, while companies with too high prices will exit because they lose their customers.

We use space in order to simplify and structure the way in which information about alternative shopping places is found. This prevents the singularity of “Bertrand-style” models, where the market share of each company is independent from history, leading to potentially huge and unrealistic fluctuations.

By doing this, one notices that the spatial dynamics can be separated from the price formation dynamics itself. This makes intuitively sense since, in generalized terms, we are dealing with evolutionary dynamics, which often does not depend on the details of the particular fitness function. We have therefore started with an investigation of a spatial competition model without prices. For this model, we have looked at cluster size distributions, and compared them with real world company size distributions. In contrast to investigations in the literature, which find log-normal distributions, we find a scaling law a better fit of our data. In the models, we find log-normal distributions or scaling laws, depending on the particular rules.

We then added price formation to our spatial model. We showed that the price, in simple scenarios, converges towards the competitive price (which is here the unit cost of production), and that it is able to track slowly varying production costs, as it should. This predicts that prices should lag behind costs of production. We indeed find this in the data of consumer price index vs. production price index for the United States since 1941.

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A Converting the aggregated census data

Non-equidistant bins The size data in the 1992 U.S. economic census comes in non-equidistant bins. For example, we obtain the number of establishments with annual sales above 25 000 k\$, between 10 000 k\$ and 25 000 k\$, etc. For an accumulated function, such as Fig. 5 (right), this is straightforward to use. For distributions, such as Fig. 5 (left), this needs to be normalized. We have done this in the following way: (1) We first divide by the weight of each bin, which is its width. In the above example, we would divide by $(25\,000\text{ k\$} - 10\,000\text{ k\$}) = 15\,000\text{ k\$}$. Note that this immediately implies that we cannot use the data for the largest companies since we do not know where that bin ends. (2) For the log-normal distribution

$$\rho(x) \propto \frac{1}{x} \exp[-(\ln(x) - \ln(\mu))^2] \quad (\text{A.1})$$

(note the factor $1/x$), one typically uses logarithmic bins, since then the factor $1/x$ cancels out. This corresponds to a weight of x of each census data point. (3) Now we have to decide where we plot the data for a specific bin. We used the arithmetic mean between the lower and the upper end. In our example case, $17\,500\text{ k\$}$. (4) In summary, say the number of establishments between s_i and s_{i+1} is N_i . Then the transformed number \tilde{N}_i is calculated according to

$$\tilde{N}_i = \frac{N_i}{s_{i+1} - s_i} \frac{s_i + s_{i+1}}{2}. \quad (\text{A.2})$$

The largest firms For the largest firms (but not for the large establishments), the census also gives the combined sales of the four (eight, twenty, fifty) largest firms. We used the combined sales of the four largest firms divided by four as a (bad) proxy for the sales of each of these four companies. We then subtracted the sales of the four largest firms from the sales of the eight largest firms, divided again, etc. Those data points should thus be seen as an indication only, and it probably explains the “kink” near 2×10^9 in Fig. 5.

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