Breakdown and recovery in traffic flow models

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Abstract. Most car-following models show a transition from laminar to "congested" flow and vice versa. Deterministic models often have a density range where a disturbance needs a sufficiently large critical amplitude to move the flow from the laminar into the congested phase. In stochastic models, it may be assumed that the size of this amplitude gets translated into a waiting time, i.e. until fluctuations sufficiently add up to trigger the transition. A recently introduced model of traffic flow however does not show this behavior: in the density regime where the jam solution co-exists with the high-flow state, the intrinsic stochasticity of the model is not sufficient to cause a transition into the jammed regime, at least not within relevant time scales. In addition, models can be differentiated by the stability of the outflow interface. We demonstrate that this additional criterion is not related to the stability of the flow. The combination of these criteria makes it possible to characterize similarities and differences between many existing models for traffic in a new way.

1 Introduction

Car traffic is not always homogeneous. For example, stop-and-go waves are a frequently observed phenomenon. Correspondingly, most traffic models show a transition from laminar to "congested" flow and vice versa. For many deterministic models, this mechanism is well understood (e.g. [1,2], see Fig. 1): For certain densities, the homogeneous solution is linearly unstable, meaning that any tiny disturbance will destroy the homogeneity and lead to another state, typically to one or more waves. For other densities, the homogeneous state may be linearly stable, but unstable against large amplitude disturbances.

In stochastic models, one would intuitively assume (see Fig. 2) that linear instability gets translated into plain instability –meaning that, for the corresponding densities, the homogeneous state breaks down immediately– and that for large amplitude instability the large amplitude instability gets translated into meta-stability – meaning that, for the corresponding densities, one has to wait some time until the noise conspires in a way that a critical disturbance is generated and the instability is triggered. This is exactly the topic for this paper, where we will demonstrate that this speculation is correct in some cases but not in others.

Recent field measurements identify additional dynamic phenomena, such as oscillations and so-called synchronized traffic [3,4]. It is under discussion in how far these additional phenomena can be explained by the above model instabilities in conjunction

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To appear in: Traffic and Granular Flow '01, edited by Y. Sugiyama and M. Schreckenberg

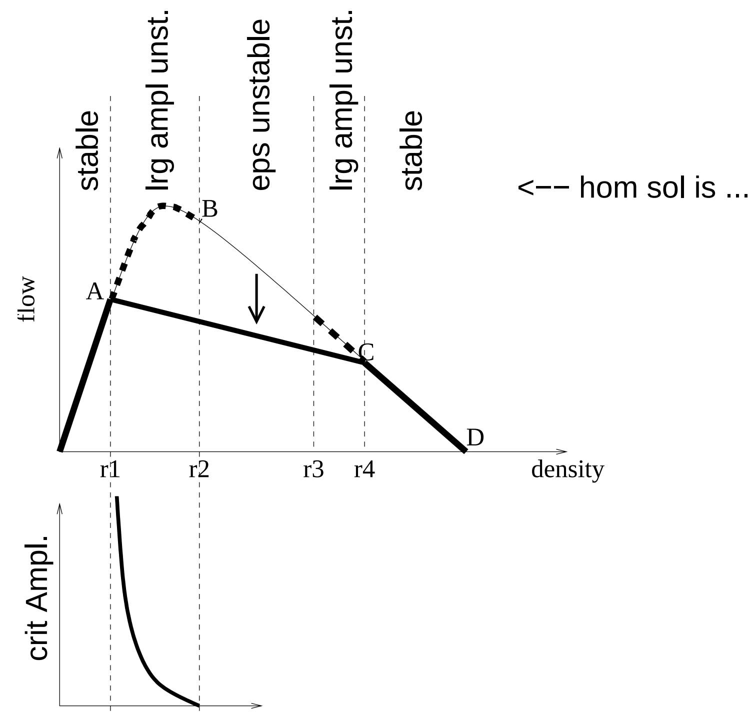

Fig. 1. Schematic fundamental diagram for deterministic models. “lrg ampl unst.” means that the homogeneous solution can be kicked into an inhomogeneous state by a large enough amplitude; the bottom plot gives a schematic graph for the necessary size of that critical amplitude. “eps unstable” means that the homogeneous solution is linearly unstable.

with geometrical constraints (such as bottlenecks) [5,6], or if additional features are necessary in the models [7]. Given this, it seems desirable to understand as much as we can about existing models.

Indeed, Krauss [8] introduces model classes which he names type I, type II, and type III. Type III refers to a viscous, syrup-like behavior without breakdown, and is not of relevance here. Type II displays jam formation, but jams have a typical size, meaning that the system is macroscopically homogeneous, and that there is no true phase transition. Type I displays true, macroscopic structure formation and therefore a first order phase transition. In this paper, we will argue that the Krauss characterization is incomplete. We will demonstrate that models can be stable or unstable at maximum flow, and that the jams can have a stable or unstable interface. The difference to Krauss in Ref. [8] is that he implicitly assumes that stable maximum flow goes together with a stable interface, and that unstable maximum flow goes together with an unstable interface. Introducing our additional characterization means that we have $2 \times 2 = 4$ different classes, instead of just I and II.

In order to demonstrate this, we will first review what expectation one has for traffic flow breakdown in analogy to a gas-liquid transition (Sec. 2). We then, in Sec. 3, de-

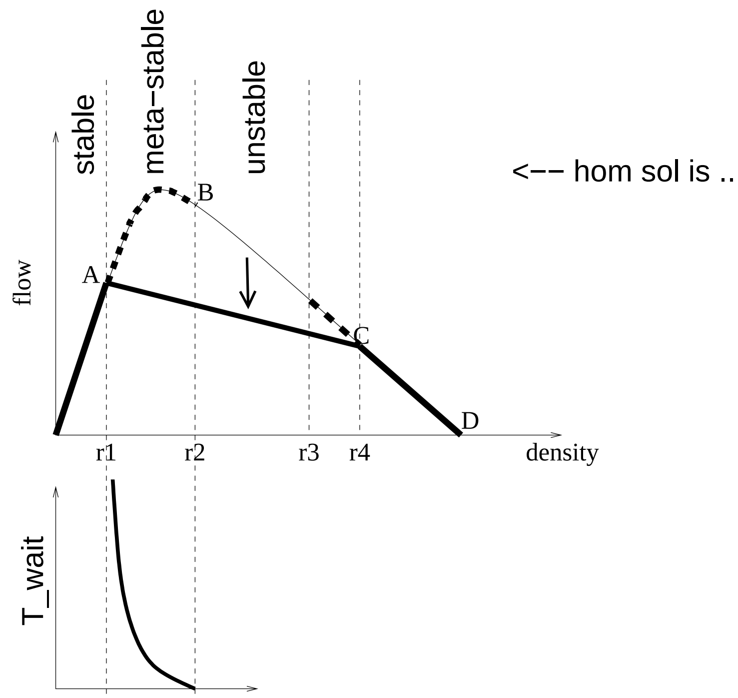

Fig. 2. Schematic fundamental diagram for stochastic models. "meta-stable" means that the homogeneous solution will be kicked into an inhomogeneous state after a certain waiting time; the bottom plot gives a schematic graph for that waiting time.

scribe the traffic model that we use. The central Sections 4 and 5 describe results with respect to transition times, and with respect to interface dynamics. Sec. 6 is a longer discussion of our results, including speculations, conjectures, and some simulation results for other models. The paper is concluded by a summary.

2 Traffic breakdown and the gas-liquid transition

The breakdown of laminar traffic, i.e. the transition from homogeneous traffic to stop-and-go waves, can be compared to a gas-liquid transition, i.e. the transition from the homogeneous gas state to the inhomogeneous gas/liquid coexistence state (e.g. [9,8,10]). As is well known, if one compresses a gas beyond a certain critical density, then it becomes super-critical, and small fluctuations will lead to droplet formation and thus into the coexistence state [11]. Similarly, we would expect for homogeneous traffic that, once compressed beyond a certain critical density, small fluctuations will lead to jam formation and thus into the coexistence state.

And conversely, one knows that all droplets vanish once the mixture is expanded beyond the critical density. Similarly, one would expect that all traffic jams vanish once the system is expanded beyond a critical density.

This leads to predictions about the statistics of jam formation and jam dissolution. Krauss [8] gives the following for the probability in a given time step that a jam starts somewhere in the system:

$$p_\downarrow \sim L \, \exp\left(\frac{\hat{\alpha}}{\langle g \rangle - g_0}\right) \,,$$

where $\langle g \rangle = 1/\rho - \ell = L/N - \ell$ is the average gap, g_0 and $\hat{\alpha}$ are free parameters, and L is the system size. N is the number of cars in the system; ℓ is the length that a vehicle occupies in a dense jam.

The above is a bulk effect; jam formation can happen anywhere in the system. Jam dissolution, in contrast, is an interface effect: A jam with N vehicles dissolves if the random numbers come out the correct way to let all N_{jam} vehicles make the "correct" type of movement. This leads to a recovery probability of $p_\uparrow \sim \exp(-N_{jam})$. Since $N_{jam} \sim L\,(\rho - \rho_c)$, we obtain

$$p_\uparrow \sim e^{-L\,(\rho-\rho_c)} \,.$$

Note that $p_\uparrow$ is larger than zero also for $\rho > \rho_c$, that is, spontaneous recovery should be possible in super-critical systems although it becomes exponentially improbable with increasing system size.

In both cases, the time to recovery would be the inverse of the above probabilities. When setting the two equations equal, one obtains the condition for a system to fluctuate back and forth between the homogeneous and the coexistence state. This would occur above ρ_c; however, for any given $\rho > \rho_c$, in the case of $L \to \infty$, $p_\downarrow$ would go to infinity while $p_\uparrow$ would go to zero, meaning that above ρ_c only the coexistence state is stable in the limit of $L \to \infty$.

3 The model

The model to be used in the following was introduced by Krauss [8]. The basic idea is that cars drive as fast as possible, but avoid crashes. Therefore, they have to choose their velocity $v \le v_{\text{safe}}$ which takes into account the braking distance $d(v)$ of the following and the braking distance $d(\tilde{v})$ of the preceding car. That means that the velocity has to fulfill $d(v) + v\tau \le d(\tilde{v}) + g$. Here, g is the space headway between the cars given by $g = \tilde{x} - x - \ell$. The braking capabilities of the cars are the same for all cars and are parametrized by the maximum deceleration b. τ is uniformly set to one throughout this paper. This safety condition can be transformed into a set of update rules as follows:

$$\begin{aligned}
v_{\text{safe}} &= \tilde{v}_t + 2\,b\frac{g_t - \tilde{v}_t}{2\,b + v_t + \tilde{v}_t} && (1)\\
v_{\text{des}} &= \min\{v_t + a, v_{\text{safe}}, v_{\max}\} && (2)\\
v_{t+1} &= \max\{v_{\text{des}} - a\epsilon\xi, 0\} && (3)\\
x_{t+1} &= x_t + v_{t+1} \,. && (4)
\end{aligned}$$

with index t counting integer time. The parameter a is the maximum acceleration, the parameter ϵ measures the degree of randomness, ξ is a random number, $\xi \in [0, 1]$, while

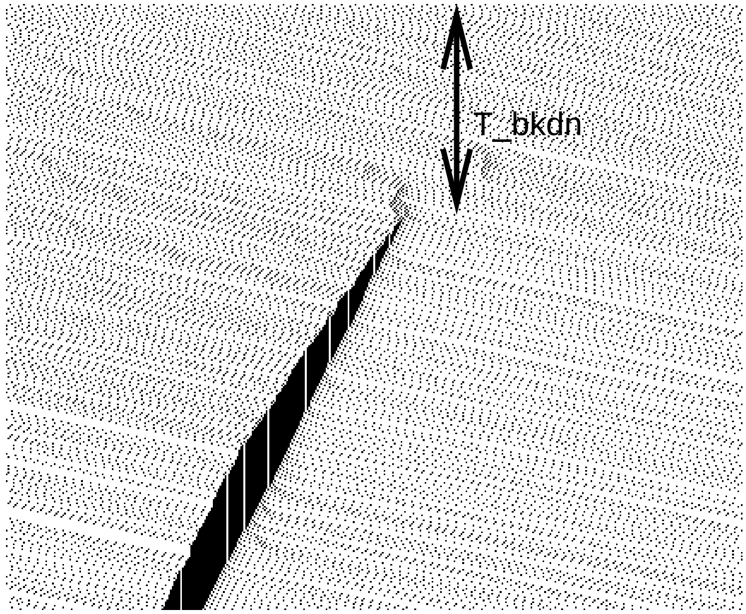

Fig. 3. Space-time plot for the breakdown time measurement. Space is horizontal; time increases downward; each line is a snapshot; vehicles move from left to right. Initially, all vehicles are lined up equidistant with the specified density. Time is measured until one vehicle in the simulation comes to a complete stop ($v_i = 0$). Once a jam is started, it typically keeps growing until inflow is reduced, either by another jam upstream, or by the effect of periodic boundary conditions.

$v_{\max}$ is the maximum velocity. We will use $v_{max} = 3$ throughout this paper. Ref. [8] discusses what our selection of parameters means in terms of real world units; let us state that our specific values have a reasonably close relation to the real world.

4 Transition times

Fig. 4 shows the breakdown and the recovery times for two different sets of parameters: $(a, b) = (1, \infty)$ and $(a, b) = (0.2, 0.6)$. Recall that a and b are the acceleration and braking capabilities, respectively. The simulations are run with a fixed number N of vehicles; different densities are obtained by adapting the system size L via $L = N/\rho_L$. The times are obtained as follows:

- **Breakdown times:** The system is started with all vehicles at equal distance $g = 1/\rho_L - 1$ and with the initial velocity taken from the laminar branch of the fundamental diagram.[1] The time is measured until the first vehicle in the system shows $v = 0$ (see Fig. 3).

[1] Annoyingly, in the transition regime, for some parameters of a and b different initial conditions lead to significantly different breakdown times. Outside the transition regime, the results are robust.

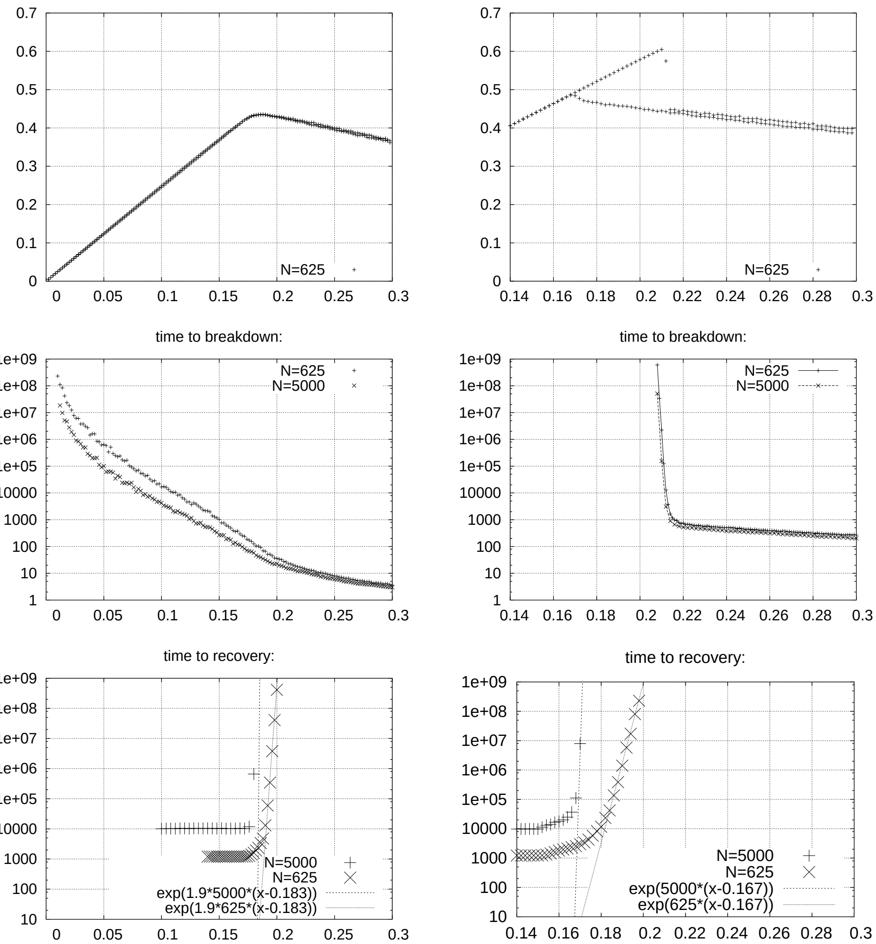

Fig. 4. Breakdown (middle) and recovery (bottom) times as function of density. Left column: $(a, b) = (1, \infty)$. Right column: $(a, b) = (0.2, 0.6)$. The straight lines in the bottom plots are proportional to $\exp(AN\rho)$, where N is the number of cars in the system, and A is a free parameter. – In the right column, we see that for $N = 5000$ there is a gap from $\rho \approx 0.17$ to $\rho \approx 0.205$, where the system is, up to 10^9 time steps, stable both against breakdown and against recovery. – The corresponding sections of the fundamental diagram (throughput as function of density; top) for $N = 625$ are given for orientation. Each value of the fundamental diagram is obtained at 5000 time steps; this is done once for homogeneous and once for jammed initial condition, resulting in two branches for bi-stable models.

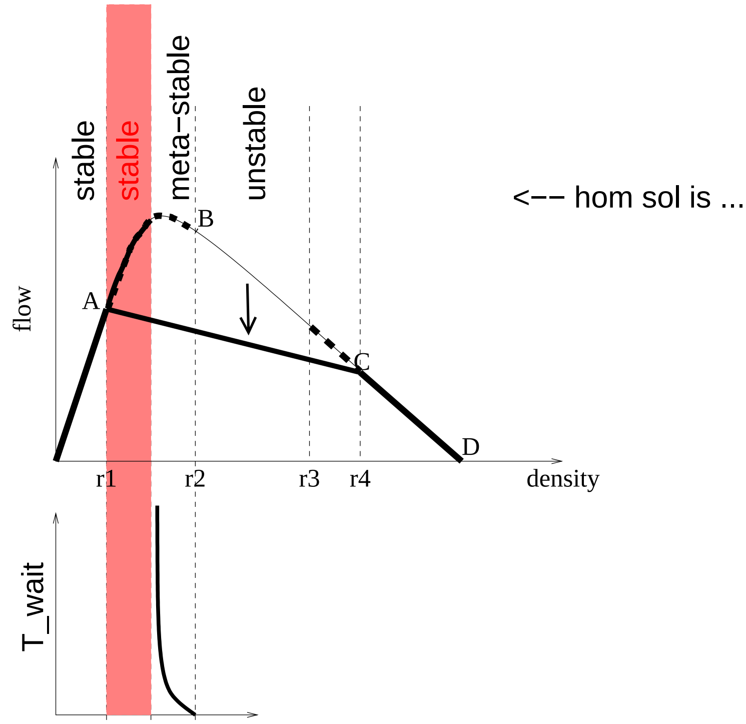

Fig. 5. Schematic fundamental diagram for stochastic models including the new regime. Compare to Fig. 2.

- **Recovery times:** The system is started with all vehicles except the leading one at distance one, i.e. gap equal to zero, and velocity zero. The time is measured until no vehicle with velocity zero is left in the system.

Each data point is an average of at least 50 runs.

One observes from Fig. 4 that, for both cases, the recovery behavior is qualitatively consistent with the gas-liquid transition picture: Above a certain ρ_c, the waiting time until recovery (i.e. until a system with jams transitions to a system without jams) shows exponential growth, which increases with system size.

Similarly, for $(a, b) = (1, \infty)$ (Fig. 4 left), the breakdown results are qualitatively consistent with the gas-liquid transition picture: The time to breakdown *de*creases both with increasing system size and with increasing density. Putting breakdown and recovery together, one obtains that for $L \to \infty$ and in equilibrium, a system with $\rho > \rho_c$ should always be in the coexistence state. For smaller L, the system can jump back and forth between coexistence and the homogeneous state.

For parameters $(a, b) = (0.2, 0.6)$, a possibly different picture emerges. Here, the breakdown times seem to diverge at $\rho^* \approx 0.2$, meaning that, for large L and possibly for $L \to \infty$, we have a density range where besides the transition from coexistence to homogeneous also the inverse transition from homogeneous to coexistence is extremely improbable. *That is, we may have a stable supercritical homogeneous phase under an update rule that includes noise.* Fig. 5 shows a schematic fundamental diagram with the new region. – It is however very difficult to get good numerical results for such fast growth as we find here: from $\rho = 0.22$ to $\rho = 0.21$, the breakdown times grow between 2 and 5 orders of magnitude, depending on the system size. Several function fits were tried out, without convincing success; for example $T \sim \exp\big((\rho-\rho_*)^{-\alpha}\big)$ (implying divergence) or $T \sim \exp\big(-\gamma\,(\rho-\rho_*)\big)$ (implying no divergence). We cannot rule out the second functional form; however, note that from Fig. 4 we have, at $N = 5000$, a gap from $\rho \approx 0.17$ up to $\rho \approx 0.21$ where both branches are stable within times of 10^9.

Also, we note that breakdown times in general do not follow well the analytical expectations. Although qualitatively the dependency is as expected (breakdown time decreasing with increasing system size), the quantitative behavior is different. The breakdown times for a given density but different system sizes do not lie on a straight line on a log-log plot (not shown), meaning that neither $T_{bkdn} \sim 1/L$ nor any other algebraic form is applicable. In contrast, for recovery times, the simulation results are at least not inconsistent with $T_{rcov} \sim \exp\big(-L(\rho-\rho_*)\big)$. One should in general note that very few system sizes were simulated; in particular, the simulations do not extend to –computationally difficult– very large systems sizes where different behavior might be found.

Why is the Krauss model with $(a, b, \epsilon) = (0.2, 0.6, 1)$ so much different from the "standard" phase transition picture, where noise will relatively quickly add up in a way that a super-critical homogeneous state will break down? We suspect that in many traffic models, because of the parallel update, noise is introduced in a special way. In particular, the amount of noise per spatial and temporal unit is bounded. In conjunction with a dynamics which dissipates noise fast enough, it make sense to obtain states which are absolutely stable under this kind of noise. It is unclear to us if the continuous variables used in the model here are a necessary ingredient or not; preliminary simulation results indicate that the same type of behavior can be obtained by a discrete model, but see Ref. [12] for a similar model where the continuousness of the variables seems to play a crucial role.

5 Interface dynamics

The nature of the transition (e.g. crossover vs. true phase transition) is however not given by the time it takes until the first fluctuation happens, but by how this fluctuation develops further, in particular, if it spreads into the rest of the system or not. In order to further understand the nature of the transition, we will now look at the dynamics of the interface between jam and outflow. That is, we start with an infinitely large mega-jam with $g = 0$ and thus $\rho = 1$ in the half space from $x = -\infty$ to zero. We collect data for the development of the density profile as a function of time. While doing that, we translate the zero of the coordinate system always to the leftmost moving car, i.e. to the rightmost car in the mega-jam which has not moved so far. To the left from this point, density is always one; in consequence, we look at the question if the interface to the right will grow in time or if it will develop a characteristic, time-independent profile.

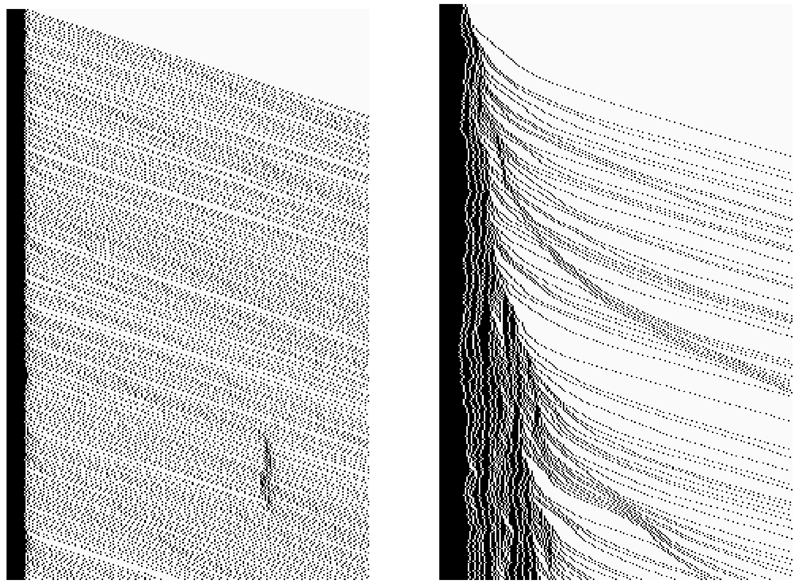

Fig. 6. Space-time plot of interface dynamics. As before, space is pointing go the right and time is pointing down. Space coordinates are translated such that the leftmost edge of the moving traffic is always at the same position. LEFT: Krauss with $(a, b, \epsilon) = (1, \infty, 1)$. Example for stable interface. RIGHT: Krauss with $(a, b, \epsilon) = (0.2, 0.6, 1.5)$. Example for unstable interface. – Note that for the left example, a and b are selected in the range typically considered unstable, while for the right example, a and b are selected in the range typically considered stable.

Fig. 6 contains space-time plots of this interface for two different systems. In the left plot, the interface is stable, whereas in the right plot, it keeps growing throughout the plot. The left plot is obtained with $(a, b, \epsilon) = (1, \infty, 1)$, which is one of the two models for which we have investigated the transition times in more detail above. A plot for $(a, b, \epsilon) = (0.2, 0.6, 1)$ looks similar (not shown). In contrast, the plot on the right with the growing interface is obtained with a larger noise amplitude, i.e. $(a, b, \epsilon) = (0.2, 0.6, 1.5)$.

In order to investigate the long-term behavior, we also plotted density profiles at different times. A stable interface is characterized by a density profile which eventually becomes stationary; an unstable interface keeps growing. Fig. 7 contains a result for the model of Fig. 6 right, i.e. with $(a, b, \epsilon) = (0.2, 0.6, 1.5)$. The plot contains density profiles at times 250 000, 500 000, 750 000, and 1 000 000. Each curve is the average of 60 runs. Clearly, the plot shows that the interface grows with time. In fact, when looking at a fixed density value, say $\rho = 0.2$, it seems that the interface width is growing linearly in time.

What this means is that *the stability of the interface is a property which is separate from the stability of the flow*. The following table lays out the resulting four cases:

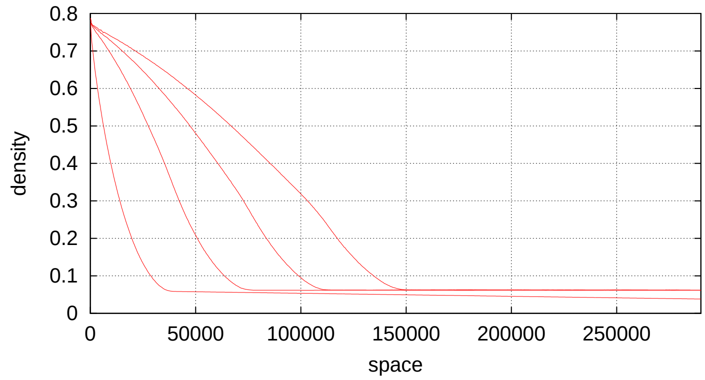

Fig. 7. Density profiles for $(a, b, \epsilon) = (0.2, 0.6, 1.5)$ at times 250 000, 500 000, 750 000, and 1 000 000. Clearly, the interface keeps growing with time. For the system on the left in Fig. 6, we obtain a completely stationary interface profile, which also does not extend far into the system (not shown).

	stable outflow	unstable outflow
stable i-face	E.g. $(a, b, \epsilon) = (0.2, 0.6, 1)$ "Krauss type I"	E.g. $(a, b, \epsilon) = (1, \infty, 1)$
unstable i-face	E.g. $(a, b, \epsilon) = (0.2, 0.6, 1.5)$	E.g. $(a, b, \epsilon) = (1, \infty, 1.5)$

The finding of 2×2 criteria goes beyond the findings of Krauss [8], who only differentiates between stable ("Type I") and unstable ("Type II") maximum flow. Krauss mentions "branching", but more or less explicitly assumes that branching goes along with unstable maximum flow. In addition, the example of Ref. [8] for "branching" ($(a, b, \epsilon) = (1, \infty, 1)$) in fact has a stable interface as demonstrated in Fig. 6.

6 Discussion and open questions

There is controversy if cellular automata (CA) models for traffic show a first order phase transition,[2] a true critical phase transition [15], or none at all [16,17]. The discussion was seriously hampered by the fact that no parameter was known to change the possibly critical behavior of the system. Our findings, with a different type of model, shed new light on this discussion. It is plausible to assume that models with an unstable outflow interface display a crossover behavior, because any phase separation in the initial conditions will spread through the system – in a finite system, there would eventually be a macroscopically homogeneous state although there would be structure on the microscopic scale. Conversely, models with a stable outflow interface will display true macroscopic phases. Since the different phases are obtained by variations of continuous parameters, it should be possible (albeit computationally expensive) to find the line in phase space which separates the two regimes.

[2] In Ref. [10], Wolf reviews evidence for and against true phase coexistence, without making a judgment.

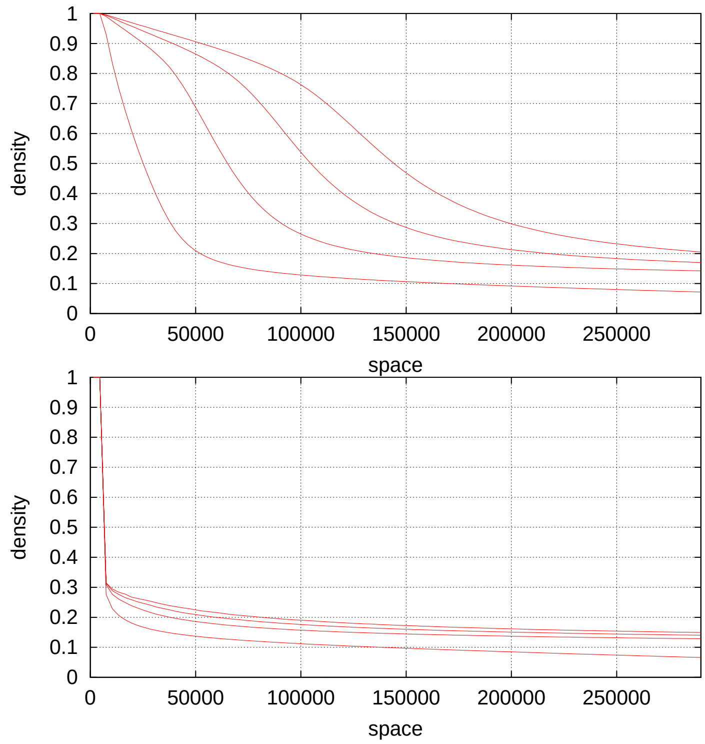

Fig. 8. Density profiles at times 250 000, 500 000, 750 000, and 1 000 000 for the stochastic traffic cellular automaton (STCA) of Ref. [13] (top) and for the slow-to-start model of Ref. [14] (bottom). Clearly, in both cases the interface is non-stationary.

Additional simulations show that the stochastic traffic cellular automaton (STCA) of Ref. [18] has indeed both an unstable outflow and an unstable interface (Fig. 8 top). The so-called cruise control limit of this model [19] also has an unstable interface but a stable outflow, although marginally so. We also tested the so-called slow-to-start model [14] and found that it has, for the parameters that we tested, an unstable interface (Fig. 8 bottom). This puts it in a class separate from Krauss type I, in contrast to the original motivation that it would display the same kind of "meta"-stability as a Krauss type I model.

Wolf, in Ref. [10], describes a so-called Galileo-invariant CA traffic model, where he observes a different type of meta-stability than the slow-to-start models. It is open into which of our four classes that model belongs.

In summary, it seems that our findings are finally the starting point of a more complete classification of the different models for traffic. Also from an engineering/applications perspective, it is necessary to solve these questions because of their consequences for real world applications. For example, the existence of stable high-flow states under noise would mean that it should be possible to stabilize these states in the real world. And an unstable outflow interface would imply different interpretations of real world data, which are typically averages over 1 minute or longer.

An implication of our findings is that, in contrast to earlier claims, outflow is not constant in STCA-type models: It is constant only outside the boundary region, which however grows to infinity. On the other hand, for both Krauss models of this paper, with $(a, b) = (1, \infty)$ and $(a, b) = (0.2, 0.6)$, and noise ϵ small enough, outflow is indeed a constant.

This implies that our theory about breakdown behavior in microscopic models needs to be revised. That theory was that there is a characteristic jam outflow, and any homogeneous solution with higher densities would be unstable against large amplitude disturbances, such as stopping a vehicle and releasing it later. For models where the outflow is not well defined this is obviously too simplistic.

In addition, it seems that also for models with stable interfaces the situation can be more complicated. Our own simulations show that, essentially, the density between jams can be "compressed" in models with continuous variables. This is not discussed further here.

7 Summary

We have demonstrated that the breakdown of the homogeneous state in stochastic traffic models is characterized by two properties: (i) stability or not of the high flow states; (ii) stability or not of the outflow interface of jams. This is different from earlier findings, where it was assumed that the two go together. This is important, since it will allow to characterize the different existing traffic models according to these properties. It should also allow to eventually settle the controversy over the nature of the transition from homogeneous to congested flow. Engineering applications should benefit from these findings by being able to pick the model type which closest reflects reality. And finally, it is an interesting physical question since we are looking at simple one-dimensional driven systems which display interesting dynamics and which can be analyzed using the methods of statistical physics.

8 Acknowledgments

We had the suspicion that the Krauss model behaved differently from the slow-to-start rules of CA models for some time. This suspicion was reinforced by discussions with D. Helbing and with D. Wolf, who reported similar observations from some of their models. Additionally, simulations performed by N. Eissfeldt helped to clarify some of the questions related to this different behavior. Computing time of about 100 000 CPU-hours on the clusters Asgard and Xibalba, both at ETHZ, is acknowledged.

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