The mechanism of flow breakdown in traffic flow models

S. Krauß,[a,b,1] K. Nagel,[c,d] and P. Wagner[a,b]

[a] German Aerospace Center (DLR), Linder Höhe, 51170 Köln, Germany
[b] Center for Parallel Computing (ZPR), University of Cologne, Weyertal 80, D-50931 Köln, Germany
[c] Los Alamos National Laboratory, Mail Stop M997, Los Alamos NM 87544, U.S.A., kai@lanl.gov
[d] Santa Fe Institute, 1399 Hyde Park Rd, Santa Fe NM 87501, U.S.A.

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Abstract

For an understanding of the congested regime of traffic flow, both the mechanisms for flow breakdown as well as the mechanisms for flow recovery after breakdown need to be considered. Although different models of traffic flow have different mechanisms of break-down, the overall mechanism that generates stable jams is common to all these models. The stability of jams, once created, is caused by a dynamics that allows outflow from jam to be less than its inflow. One way to achieve this is to reduce the acceleration of cars upon leaving a jam (slow–to–start). This explains the capacity drop of the fundamental diagram and makes the transition from free flow to congested flow similar to a phase transition of first order.

Break-down requires the existence of two types of solution: a homogeneous and a jam solution. If a model has these two solutions, break-down occurs by losing the stability of the homogeneous solution upon increasing the density. For deterministic models a small perturbation then leads to the transition from free flow to jam. Stochastic models differ from this picture in that they may break down at smaller values of the density. If the model has the slow–to–start-feature, this is enough to stabilize the jam solution, provided the density is large enough. This article describes how different models of traffic flow fit into this general picture.

1 Introduction

Recently, simplified microscopic models of traffic flow have spawned a number of new research results over the phenomenological description of traffic, and the modelling of traffic flow in general. This contribution deals with the mechanisms of traffic flow breakdown and recovery. It gives an overview about some of the microscopic modelling approaches, especially in view of recent empirical findings.

The empirical findings, that have emerged over the past thirty years, are very well summarized and extended according to [1]: (1) Traffic jams, once created, are fairly stable and can move without major changes in their form for several hours against the flow of traffic, (2) the flow out of a jam can be as low as 2/3 of the maximally possible flow, therefore creating a branched fundamental diagram with a meta-stable high-flow state (laminar flow phase;

[1]Current affiliation: Bain & Company, Munich, Germany

resulting in the typical reverse λ-shape), and (3) there exists a third kind of traffic-flow states, which has been called synchronized traffic.

Focusing on traffic jams, there are two different questions to be addressed: the creation of jams and their stability. The creation mechanism is different for different models, while the mechanism that is responsible for their stability is universal and very simple: when the jam escape time is larger than the "reaction" time in moving traffic, jams are readily stabilized once they are created (B. Kerner, personal communication, [2]).

The reason behind this consists of two arguments:

- For driving to be crash free, under homogeneous conditions the time headway needs to be larger than the reaction time. For example, assume two identical vehicles driving with the same speed. Let the first vehicle initiate an emergency braking with a certain emergency braking deceleration profile $b(v)$ at time t_0. Because of reaction delay, the following vehicle initiates an emergency braking at time $t_0 + \tau$, where τ is the "reaction" time (not strictly a human reaction time, but a delay time caused by human and mechanical delays). Since the vehicles are assumed to be identical, the braking of the second vehicle will also follow the deceleration profile $b(v)$. One can now see that, if the time headway at t_0 was smaller than τ, then the second vehicle will crash into the first one. Thus, in short, time headway needs to be larger than "reaction" time; and in practice they will be similar because people optimize their behavior towards the physical limits of the system.

- Now, the inverse of the time headway is the rate with which vehicles get added to the jam, while the inverse of the escape time is the rate with which vehicles leave the jam. Obviously, if the escape time is larger than the time headway, then the jam grows.[2]

Thus, again, jams are stable when the typical escape time between vehicles leaving a jam is larger than driver's reaction times in homogeneous flow.

Obviously, any model that matches the conditions (1) and (2) above has to have this feature. The different models differ in the mechanisms that lead to flow breakdown. The main difference is between deterministic and stochastic models, and this is what is to be described next. Most of the descriptions in this paper technically refer to "closed systems", i.e. idealized "traffic in a closed loop" where vehicles neither enter nor leave the system. This is done for the purpose of an easy exposition; all arguments can be extended to open systems. Nevertheless, open systems may behave differently than closed ones. This can be seen for the CA–models of section 5, where analytical results for open systems exist (e.g. [3]).

2 The deterministic picture

For deterministic models, the mechanism of flow breakdown is as follows (see Fig. 1):

[2]Technically, one would have to correct the time headway and the escape time for the backwards movement of the jam wave. Yet, since this speed is roughly the same for the upstream and the downstream front of the jam, it is irrelevant for this argument.

(i) There is a range of densities $\rho_{l1} < \rho < \rho_{l2}$ where the homogeneous solution is linearly instable, which means that laminar traffic breaks down to start-stop traffic because of the tiniest disturbance, for example caused numerical imprecisions.

(ii) There is usually another, wider region bounded by ρ_{n1} and ρ_{n2} (i.e. $\rho_{n1} < \rho < \rho_{l1}$ and $\rho_{l2} < \rho < \rho_{n2}$) where the models are bi-stable, i.e. both the homogeneous and the start-stop solution are linearly stable. A certain critical amplitude "kicks" the system out of its homogeneous state. In the laminar phase, the size of the amplitude goes to zero when approaching ρ_{l1} from below, and to infinity when approaching ρ_{n1} from above.

(iii) Outside of ρ_{n1} and ρ_{n2}, only the homogeneous solution is stable.

In the regime $\rho_{n1} < \rho < \rho_{n2}$, the system can be described as a mixture of regions where cars are jammed and regions where cars drive freely, i.e. where the system is in its homogeneous state. This is exactly the signature of a first-order phase transition connecting a state with all cars moving freely and all cars moving (or standing) in a jam.

ρ_{n1} is actually just the density of the outflow from a macroscopic jam. It is easy to see why: Assume laminar traffic has a density ρ larger than ρ_{n1}, and assume it is forced to form a macroscopic jam, for example by stopping a vehicle for several seconds. Then the inflow into this jam is $q_{in} = v(\rho)\,\rho$, and the outflow is $q_{out} = v(\rho_{out})\,\rho_{out}$. As long as $\rho \approx \rho_{out}$, we have that $v(\rho) \approx v(\rho_{out})$. Therefore, $q_{in} > q_{out}$ when $\rho > \rho_{out}$, and then the jam is stable, meaning that any $\rho > \rho_{out}$ is instable against a macroscopic disturbance, and therefore $\rho_{out} = \rho_{n1}$. [3]

The difference between ρ_{l2} and ρ_{n2} is not important for the purposes for the present paper, but it is probably important for the explanation of synchronized traffic [1, 5, 6].

The above description follows a similar analysis by Kerner and Konhäuser for fluid-dynamical equations for traffic flow [7].

3 The stochastic picture

When adding noise, the formerly bi-stable regime is no longer strictly bi-stable, but there exist transition probabilities from one regime to the other.

- **Transition from laminar to start-stop.** Consider a density where the corresponding deterministic model is in the bi-stable regime, at the laminar branch (see Fig. 2). A jam can form when enough nearby vehicles drive, due to noise, more slowly than they have to. This happens deterministically when $\rho = \rho_{l1}$. For $\rho < \rho_{l1}$, following a simple consideration (see section 5 and [2]), the probability for a car to be stopped, $P(0)$, can be approximated as

$$P(0) \propto \exp\left(\frac{\alpha\,\eta}{\rho_0 - \rho}\right) ,$$

[3]For a technically correct description, one actually needs to take the movement of the interfaces (jam fronts) into account. As long as jams are compact (see [4]), the condition $q_{in} > q_{out}$ remains the same.

where η denotes the strength of the noise. ρ_0 is roughly equal to ρ_{l1}. Since in a large system there are more spatial regions where this can happen, the systemwide probability of breakdown is $P \propto L\,P(0)$, where L is the system size. That is, higher noise and large system sizes increase the probability of a breakdown *somewhere* in the system. For example, in a system twice as big the probability of break-down is twice as high. Also, the probability of break-down increases when approaching ρ_{l1} from below; as already said, P becomes one even before $\rho = \rho_{l1}$.

- **Transition from start-stop to laminar.** Assume, for the sake of simplicity, that there is only one jam in the system, and remember that for our explanations we assume the system to be closed. The number of vehicles in that single jam is proportional to $(\rho - \rho_{n1}) \cdot L$. In order to get into the laminar regime, *all* vehicles in the jam need to make "fast" accelerations when leaving the jam. Since the choice between "fast" and "slow" accelerations is given by noise, the probability that all vehicles do a "fast" acceleration is $P \sim \eta^{(\rho-\rho_{n1})\cdot L}$. That is, higher noise increases the probability of recovery, but larger system sizes (or larger jams) *decrease* it. For example, if $\eta = 10\%$, having a jam twice as big means a tenfold reduced chance of recovery.

The result of this is that, for small noise and large system sizes, the probability of break-down is much bigger than the probability of recovery, since the probability of break-down increases with system size, and the probability of recovery *de*creases with system size. For that reason, such systems practically never recover by themselves, but only if demand vanishes, for example due to the end of the rush hour. Such a description of break-down in terms of probabilities is also consistent with observations [8]. In the following sections, a more detailed account on the mechanisms of break-down will be given.

4 Continuous time and continuous space

Car-following models that use continuous space and continuous time are generally of the form

$$\dot{v}_n = f(v_n, \Delta x_n, \Delta v_n)$$

where v_n is the velocity, $\dot{v} = dv/dt \equiv a$ is its time derivative and thus the acceleration, $\Delta v_n = v_{n-1} - v_n$ is the velocity difference and $\Delta x_n = x_{n-1} - x_n$ is the distance to the car ahead [9, 10].

4.1 The Optimal Velocity Model (OVM)

A model that is in the tradition of [11] is the so called optimal velocity model (OVM) [9] (see also [12]). Its acceleration equation reads:

$$\dot{v} = \alpha \cdot (V(\Delta x) - v) \tag{1}$$

where the index n is left out for convenience, $V(\Delta x)$ is a sigmoid function that has the features $V(\Delta x) \to v_{max}$ if $\Delta x \to \infty$, and $V(\Delta x) = 0$ at a certain distance Δx. This model has been analyzed into great detail in [9]. It is an example for a model that displays the deterministic picture described in Sec. 2. For small densities, only a homogeneous solution exists, where all cars drive with the same velocity. At a certain density ρ_{l1}, this homogeneous solution becomes linearly unstable to be exchanged by a solution where the system has several jams that are separated by regions of free-flow. The density ρ_{out} of outflow from such jams is smaller than ρ_{l1}, and for that reason a large enough disturbance can move traffic from the laminar into the start-stop regime already for densities below ρ_{l1} (but larger than ρ_{out}). Again, for that reason $\rho_{out} = \rho_{n1}$.

This model is structurally stable, in the sense that small changes in the model equation do not change its general behaviour. For example, the explicit introduction of a delay time in eq. (1), i.e.

$$\dot{v}(t) = \alpha\,\left[V(\Delta x(t-\tau)) - v(t)\right]\,,$$

does not change much. Although the delay changes the *microscopic* picture in that it changes the stable fixed point at $v = V(\Delta x)$ into a limit cycle for a certain range of velocities, the *macroscopic* behaviour of the model does not change much, as indicated by numerical simulations. In general, this model has the disadvantage that it is not completely crash-free, a problem that is aggravated when using the time delay, and it displays very large accelerations.

4.2 Classic Car-following Models

Interestingly, the "classic" car-following model family [10]

$$\dot{v}(t) = \alpha \frac{v^l(t)}{[\Delta x(t-\tau)]^m} \Delta v(t-\tau) \qquad (2)$$

(dropping again the car index to simplify notation) is a noteworthy exception from the transition mechanism described in Sec. 2. In some sense, the problem is easy to recognize since there is no preferred velocity for any given density: homogeneous solutions exist for arbitrary spacings and velocities as long as everybody has the same velocity.

But let us first look at the simplified situation of one vehicle following another vehicle whose velocity $v_0 > 0$ [13]. This system has three different limit sets, i.e. attraction states for the dynamic eq. (2): (i) $\Delta v = 0,\ \forall\, v,\ \forall\, \Delta x$, (ii) $v = 0$, and (iii) $\Delta x \to \infty$. In a certain sense, there is a fourth set $\Delta x = 0$, meaning that certain combinations of parameters and initial conditions do lead to a crash between the two cars. This true if $m < 1$ and $v(0)$ is large.

(i) means that any space headway is a stable solution as soon as the follower has the same speed as the leader. (ii) means that the second vehicle having no speed at all is a fixed point of eq. (2). The limit set (iii) with $\Delta x \to \infty$ may look surprising at first, but has a simple explaination: if the initial distance to the car ahead is too large, the following car simply is not able to follow the car ahead since the acceleration is not high enough; eventually, the follower will be left so far behind that, because of the large Δx, the speed of the follower will not change any more.

All this can be seen in the phase portrait in Figure 3. Two different informations are given: (i) stability of the fixpoint against small disturbances; and (ii) trajectories of the dynamics of eq. (2) for different initial conditions.

Fixed points in the dark region are instable and thus initiate exponentially growing oscillations except for $\Delta v = 0$. Values in the light region have oscillatory but damped solutions, i.e. small disturbances of $\Delta v = 0$ lead to an exponentially decreasing oscillation for the velocity of the follower and eventually to the same velocity as the leader. Values in the white region are stable, i.e. disturbances to a fixpoint do *not* lead to oscillations.

Black lines give trajectories in the phase space for different initial conditions. Note, that Figure 3 actually is a projection of the infinite dimensional phase space of eq. (2) onto the $(\Delta x, v)$–plane. It is infinite dimensional because of the delay τ; in order to solve eq. (2) as an initial value problem, whole functions $\Delta x(t), v(t)$ in the time-interval $[-\tau, 0]$ must be given. The velocity of the leader is assumed to be constant.

The trajectories in Figure 3 have been integrated by using a highly sophisticated numerical code [14] that allows for adaptive step-size control in a delay differential equation. For the parameters used in Figure 3, no collisions occur. By lowering the velocity of the leading vehicle, collisions appear, which is the reason why this set of equation could not be integrated for more than a small number of vehicles only or for the special situation of N cars following a leading car.

As can be seen in Figure 3, initial conditions with a small headway Δx and large enough velocity converge towards $\Delta v = 0$. Initial conditions with large headways and small velocities have a tendency to never pick up enough speed and thus to get "stuck" at velocities smaller than the velocities of the leader.

For $\tau = 0$ this phase-portrait can be computed analytically by solving the differential equation for $dv/d\Delta x$. For example, for $l = 1, m > 1$ (Figure 3), it can be shown that there exist initial conditions where $v(t) \to v_\infty < v_0$ for $t \to \infty$. This happens for any car whose initial velocity is smaller than $v_0 \exp[\alpha(\Delta x)^{1-m}/(1-m)]$.

Note that this is not in contradiction to the results of traditional car-following theory; the difference is just that the traditional theory is mostly concerned with small perturbations to fixed points whereas Figure 3 looks at behavior starting far away from possible fixpoints.

The stability of the line of fixed points $\Delta v = 0$ which are related to car-following can be determined analytically by applying a linear stability analysis with respect to $\Delta v = 0$. After linearization, the stability of this limit set can be analyzed further by inserting the ansatz $\Delta x \propto \exp(-\lambda t)$. Since the whole procedure is just a corollary of the analysis done in [13], only the main results will be stated here. The linearized model is just the linear model in [13]:

$$\dot{\delta v}(t) = \alpha \frac{v_0^l}{g_0^m} \delta v(t - \tau)$$

with a different pre-factor. (Here, g_0 denotes the distance to the leading car.) Therefore the steady state is (i) stable (i.e. overdamped) if $\alpha \tau v_0^l / g_0^m < 1/e$, it is (ii) oscillatory damped if $\alpha \tau v_0^l / g_0^m < \pi/2$, and (iii) it is unstable else.

When looking at more than two cars, the instability of the model gets worse. Only when starting from a homogeneous state, the system can be integrated without running into problems. When starting with some of the vehicles standing while the others are driving, all simulations end catastrophic: cars collide, since there is no guard against crashes. It was not possible to integrate a system with more than 50 cars for more than a few seconds until a crash happens. Different values for l, m have been used, e.g. $m = 0.8, l = 2.8$, or as measured from car-following experiments, $m = 0.55, l = 1.1$ [15]. Even a number of regularization attempts did not change this behaviour, such as (i)

$$v^l \;\to \mathrm{sign}(v)|v|^l$$

or (ii)

$$1/\Delta x^m \begin{cases} = C & \text{if } \Delta x < g_c \\ = 1/(\Delta x)^m & \text{else} \end{cases}$$

that would allow for small violations of $v \geq 0$ and $\Delta x \geq 0$. The problem consists of the fact that for $m > 1$ crashes happens if cars have to stop behind other cars, while crashes occur for $m < 1$ if cars reaccelerate after a jam. This happens because the cars accelerate so fast that they pass the fixed point $\Delta v = 0$ and then could not manage to decelerate before they have reached the car ahead. The only exception was found by allowing for different m depending on the sign of the velocity difference and by using very small values of the delay time τ. But note that strictly speaking, this is a different model. However, in this case the profiles $v(x)$ evolve toward a homogeneous state.

Our results can be summarized as follows (see, e.g. [16] for similar conclusions):

- There is no acceleration without lead vehicle. Therefore, an additional acceleration term is needed for realistic applications. A possible acceleration term would be $\dot{v} \propto v_{max} - v$, to be added to the Δv term.
- However, the model is structurally unstable. For example, adding the above acceleration term $\beta(v_{max} - v)$ leads to a different model; in this case, the limit set $\Delta v = 0$ vanishes. Models of this type are very unlikely to describe real behaviour, since any small change in the equations leads to drastic changes in the dynamical behavior.
- There is no preferred distance, because of $\dot{v} \propto \Delta v$; any state where Δv vanishes will do. Therefore it is very unlikely to observe spontaneous structure formation which is so familiar in the other models discussed in this paper.

In summary, one can maybe say the following: The car-following models by Herman et al. describe the *onset* of instabilities in a homogeneous situation, and how this is influenced by the parameters: Small sensitivity α, small "reaction" time τ, small velocity v_0, and large spacing g_0 all suppress instabilities. In that very limited sense, these models are consistent with the car following models from Sec. 4.1: Both for very large spacing and for very small velocity, stable homogeneous (laminar) solutions exist. Yet, the traditional car-following models are incapable to describe traffic *beyond* the onset of instabilities since they lack a mechanism that limits oscillations to realistic values – i.e. to values limited by the acceleration and braking capabilities of vehicles.

5 Discrete time and discrete space

In the last section, we described car-following models that were continuous both in time and in space. Intuitively, it seems that these should be the only useful models, or at least the most realistic ones, since reality is clearly continuous both in time and in space. However, computer implementations often are "time-stepped" and thus effectively coarse-grained in time, for example in one second increments. And in fact much progress in recent years has been made by using models that take this approach one step further: they also coarse-grain space. These models are often called cellular automata (CA).[4]

In these models, space is typically coarse-grained to the length a car occupies in a jam ($\delta x = 1/\rho_{jam} \approx 7.5\ m$), and time typically to one second (which can be justified by reaction-time arguments [2]). One of the curious side-effects of this convention is that space can be measured in "cells" and time in "time steps", and usually these units are assumed implicitely and thus left out in the equations. A speed of, say, $v = 5$, would mean that the vehicle travels five cells per time step, or 37.5 m/s, or 135 km/h, or approx. 85 mph.

CA models are typically rule-based (instead of equation-based, such as eq. 2); a simple CA model can for example be as follows [17, 18, 19]:

- **Car-following rule:** $v(t-1/2) = \min[v(t-1)+1, gap(t-1), v_{max}]$
- **Randomization:** $v(t) = \begin{cases} \max[v(t-1/2), 0] & \text{with probability } p_{noise} \\ v(t-1/2) & \text{else} \end{cases}$
- **Moving:**

$$x(t) = x(t-1) + v(t)$$

$t-1$ and t here refer to the actual time-steps of the simulation; $t - 1/2$ denotes an intermediate result used during the computation.

The first rule describes deterministic car-following: try to accelerate by one velocity unit except when the gap is too small or when the maximum velocity is reached. Gap is the distance to the car ahead, $g = \Delta x - \ell$, where Δx is the front-bumper-to-front-bumper distance, and ℓ is the length the vehicle ahead will occupy in a jam.[5]

The second rule describes random noise: with probability p_{noise}, a vehicle ends up being slower than calculated deterministically. Without further discrimination, this parameter simultaneously models effects of (i) speed fluctuations at free driving, (ii) over-reactions at braking and car-following, and (iii) randomness during acceleration periods.

Despite somewhat unrealistic features on the level of individual vehicles, these models describe aspects of the macroscopic behavior correctly. In addition, for certain simpler CA models it can be shown that they have fluid-dynamical limits that turn out to be Lighthill-Whitham equations [19, 20], which is nice to know for scientific consistency.

[4] We use the term "coarse-grained discrete" in order to distinguish from the "fine-grained discreteness" that floating point numbers in computers have. In this paper, we will not distinguish between the fine-grained discreteness of floating point numbers and the continuity of real numbers.

[5] The use of Δx is not entirely consistent in the literature. Usually, it denotes front-bumper-to-front-bumper distance, but Herman et al (see [10]) note correctly that one can "cut out" the lengths of the cars and not change anything in the mathematics of their model. This is not longer true for other models, and so we use gap g for a quantity that we assume to be zero in a jam, whereas for Δx, we leave this open.

5.1 Deterministic CA 184

A simple deterministic CA model can be obtained when in the above rule-set the randomization rule is dropped. In that situation, all initial conditions eventually lead to one of two regimes, depending on the overall density in the system:

- **Laminar traffic:** Vehicles move with velocity v_{max}, and the gap between vehicles is either v_{max} or larger. In consequence, the flow in this regime is is $\langle q \rangle_L = \langle \rho \rangle_L \, v_{max}$. This happens when the density in the closed system is smaller than $\rho_c = 1/(v_{max}+1)$. $\langle . \rangle_L$ refers to averages taken over the whole system of size L.[6]

- **Congested traffic:** If the density $\langle \rho \rangle_L$ is larger than ρ_c, not all vehicles can move with maximum speed. The system goes to a state where the velocity of a vehicle is always the same as its gap (see Fig. 5). In consequence, the average speed is

$$\langle v \rangle_L = \langle gap \rangle_L = \frac{1}{\langle \rho \rangle_L} - 1$$

  and the average flow thus is

$$\langle q \rangle_L = 1 - \langle \rho \rangle_L \ .$$

  One will recognize Greenshield's model $v = (1/\rho) - (1/\rho_{jam})$ and $q = 1 - (\rho/\rho_{jam})$ with $\rho_{jam} = 1$ due to the cellular structure of the model.

- The two regimes meet at

$$\rho_c = 1/(v_{max} + 1) \text{ and } q_c = v_{max}/(v_{max} + 1) \ ; \tag{3}$$

  this is also the point of maximum flow.

What all this means is that there is a maximum possible microscopic driving speed, $v = \min[gap, v_{max}]$, and that the traffic adjusts in a way that vehicles always drive with this maximum possible speed. In consequence, also the macroscopic traffic flow follows exactly this maximum microscopic driving speed. With respect to Fig. 1, this means that the points A and C and thus the connecting line between them do not exist. Somewhat more precisely, B moves up to the maximum of the fundamental diagram, and A moves up to B. Regarding C, in some sense it becomes spread out over the whole decreasing branch of the fundamental diagram, since in this regime, *all* configurations are marginally stable, including completely laminar configurations and arbitrary start-stop-wave configurations.

5.2 Slow-to-start rules for deterministic CA models

In the introduction, we mentioned that, in order for jams to be stable, the reaction time and thus the minimum time headway need to be smaller than the "escape" time. The CA

[6] Remember that for the purposes of this paper, we concentrate on closed systems, i.e. "traffic in a lon loop". Local time averages in open systems, which come out of field measurements, can be derived from these arguments, but make the explanations more cumbersome and will for that reason mostly be omitted in this paper.

from Sec. 5.1 violates this rule because both times are exactly the same (which explains why the congested regime is exactly at the margin between stability and instability). A way to obtain models that represent this aspect of the dynamics correctly is the use of so-called slow-to-start rules [21, 22]. One simply slows down acceleration from low speeds, for example by saying

- if $(v < 1)$ then

$$v(t-1/2) = \min[v(t-1)+1, gap(t-1)\underline{-1}, v_{max}]$$

- else

$$v(t-1/2) = \min[v(t-1)+1, gap(t-1), v_{max}] \ ,$$

where the underlined part is the important difference. Here, a vehicle with speed less than one needs to wait until the gap to the vehicle ahead is at least two before it is allowed to move – in a typical outflow situation from a dense jam, this will take two time steps, thus making this "escape time" larger than the minimum headway time.

The result is the typical "reverse λ" shape in the fundamental diagram – a laminar branch that goes from zero up to ρ_c and q_c (point B in Fig. 1) as defined in Eq. (3), and a start-stop branch that goes from ρ_{n1} and q_{n1} (point A) to $\rho_{jam} = 1$ and $q = 0$ (point D). As explained earlier, ρ_{n1} is given by the outflow from a macroscopic jam. In the deterministic slow-to-start model, it is thus [22]

$$\rho_{n1} = \frac{1}{2v_{max}+1} \ ,$$

which is indeed smaller than $\rho_{l1} = \frac{1}{v_{max}+1}$.

5.3 "Traditional" stochastic CA models

The complete stochastic CA model as introduced at the beginning of this Section 5 needs a description in terms of probabilities rather than in terms of deterministic transitions between regimes. The minimum gap during free driving is v_{max}, whereas the typical escape gap is about three times as big (assuming $p_{noise} = 0.5$). Thus, this model in principle qualifies according to the criteria layed out in the introduction. Yet, in practice the noise is so strong that the laminar branch of the fundamental diagram is totally unstable – or in other words: noise is so strong that completely laminar traffic at *any* density breaks down after very few time steps. Now, if the overall density is low enough, the resulting disturbance will "sort itself out", thus having caused nothing more than a short annoyance for the driver. If the inflow density is high enough, the resulting disturbance could lead to a macroscopic jam – except that under those conditions, traffic in the inflow region would also break down.

5.4 Stochastic CA models with slow-to-start rule

It is obviously possible to add a slow-to-start rule as in Sec. 5.2 also to stochastic CA models [22, 23]. A possible set of rules would be:

- **Car-following rule:**

$$v(t-1/2) = \begin{cases} \min[v(t-1)+1, gap(t-1)-1, v_{max}] & \text{if } v(t-1)=0 \\ \min[v(t-1)+1, gap(t-1), v_{max}] & \text{else} \end{cases}$$

- **Randomization:** $v(t) = \begin{cases} \max[v(t-1/2), 0] & \text{with probability } p_{noise} \\ v(t-1/2) & \text{else} \end{cases}$

- **Moving:**

$$x(t) = x(t-1) + v(t)$$

The result is the maybe so far most convincing description of car following using CA models (Fig. 2). Traffic can remain laminar up to a density ρ_{l1} for long times (which get shorter when approaching ρ_{l1}); once the traffic breaks down, the ouflow is at a density ρ_{n1} significantly smaller than ρ_{l1}.

5.5 Linear vs. large-amplitude instability in CA models

In CA models, the notion of an "arbitrarily small" disturbance does not make sense: disturbances are always at least of the size of the coarse graining. For that reason, in the slow-to-start model (and in the so-called cruise-control model [4], which has not been described here), the equivalent of the linear instability does not exist. What happens instead is that the *amplitudes* for break-down from the continuous models get translated into *probabilities* for break-down in the CA models.

Take, for example, the slow-to-start model at a bi-stable density, i.e. $\rho_{out} < \rho < \rho_c$. Picking an arbitrary vehicle and reducing its velocity by one can initiate a jam only if the next vehicle follows with $gap = v_{max}$. The probability that this is the case increases when approaching ρ_c from below; at ρ_c it is exactly one. Thus, when picking an *arbitrary* vehicle in a system with density ρ, the probability $p_{bd}(\rho)$ of causing a break-down by this is zero when $\rho < \rho_{out}$, it is one when $\rho \geq \rho_c$, and it is a monotonically increasing function of the density for $\rho_{out} < \rho < \rho_c$.

Thus, in short, the concept of critical amplitudes in deterministic continuous models gets replaced by probabilities in stochastic CA models – a large critical amplitude means a small probability of break-down and vice versa.

Let us now assume a certain level of given noise – say, in each time step a car may be slowed down from maximum speed with a small probability η. It turns out that, in each time step, the systemwide probability of a break-down can be approximated as [2]

$$P_{bd}(\rho) \sim L \, \exp[C \, \eta \, (\rho_0 - \rho)] \; ,$$

where ρ is the system-wide density, L is the system size, C is a constant, and ρ_0 is roughly [2] $1/(v_{max}+1) = \rho_{l1}$. The typical waiting time to one break-down in such a system is

$$T_{bd}(\rho) = \frac{1}{P_{bd}(\rho)} \sim \frac{1}{L} \, \exp[-C \, \eta \, (\rho_0 - \rho)] \; .$$

This means in effect a trade-off between the density and the probability of break-down. For example, let us assume one is ready to accept a certain probability of break-down, which could for example be smaller than the inverse of the length of the rush period (otherwise one can be practically certain that break-down will happen). If one has a certain noise level η caused by human driving, this means that one would have to keep the density ρ below a certain value. *The longer the rush period, the lower this density would have to be.* If, however, one would be able to reduce η (for example by using automated driving systems), one would be able to accept a higher A_c and thus a higher density. This theoretical picture is also consistent with field observations [8].

6 Discrete time and continuous space

In this paper, we started with models that are continuous in space and time since they seemed best suited to describe reality. We then moved to models that are coarse-grained discrete both in space and in time. Because of their simplicity, these models are best suited for understanding certain effects and phenomena, but they have their shortcomings especially when they need to be calibrated to reality. There is a third class of models that sits "in between" the other two clases: the class of models that are continuous in space but coarse-grained discrete in time.

Such models with continuous space but coarse-grained time [24, 2, 25, 26, 27] can be understood as either discretizations of the car-following differential equations in Sec. 4, or as state-continuous pendants of the cellular automata (CA) models in Sec. 5. From a practical point of view, they are more suitable for applications than the CA models because they are numerically as efficient as the CA models and are very easy to calibrate. They are also easier to implement than models with continuous time since they explicitely use coarse-grained time in the model formulation and thus allow to use a coarse-grained time-stepped update scheme.[7]

Obviously, a multitude of models is possible in this class – as in all of the other classes. Yet, in this section we want to concentrate on a single model, a model described by Krauß [2, 25]. The two main reasons why we want to do this are (i) the model is particularly well understood, and (ii) the model makes an explicit attempt to be rooted more in microscopic reality than the models discussed so far. The approach starts from a very simple consideration of the braking distances, i.e. from the observation that my own braking distance plus the distance I drive until I react should be smaller than the braking distance of the car ahead plus the space in between the two vehicles. Formally, this yields

$$d(v_n) + v_n \tau \leq d(v_{n-1}) + g_n \ , \qquad (4)$$

where $d(v_n)$ is the braking distance of the n-th car moving with speed v_n, τ is the reaction time and g_n is the distance to the car ahead. Note that g_n here denotes the "gap" in the same way it was used in the cellular automata models, i.e. $g_n = \Delta x_n - \ell$, where Δx_n is the front-bumper-to-front-bumper distance, and $\ell = 1/\rho_{jam}$ is the space the vehicle uses up in a jam. In consequence, $g_n = 0$ in a jam.

[7] This is not to say that implementations of quasi-continuous time, e.g. using event-driven schemes, is impossible. It is difficult to implement, though, in part because of the time delay, and, possibly for that reason, currently not much used in transportation simulations.

From this, a simple collision-free update scheme can be derived:

$$v_{safe} = v_{lead}(t) + b\frac{g(t) - v_{n-1}(t)\tau}{\bar{v} + b\tau} \tag{5}$$
$$v_{det} = \min[v(t) + a, v_{safe}, v_{max}] \tag{6}$$
$$v(t+1) = \max[0, v_{det} - \eta\, a\, \xi] \tag{7}$$
$$x(t+1) = x(t) + v(t+1) \tag{8}$$

The meanings of the different terms will be explained in the following:

- The first term calculates the maximum "safe" velocity. This velocity is obtained by using Eq. (4) as an equality, and expanding it around the stationary state $v_n = v_{n-1} = \bar{v}$:

  $$d(\bar{v}) + d'(\bar{v}) \cdot (v_n - \bar{v}) + \ldots + \bar{v}\,\tau + (v_n - \bar{v})\,\tau = d(\bar{v}) + d'(\bar{v}) \cdot (v_{n-1} - \bar{v}) + \ldots + g_n\ .$$

  After noting the kinematic relation

  $$d'(\bar{v}) \equiv \frac{d}{dv} d(\bar{v}) = \frac{\bar{v}}{b(\bar{v})}$$

  and rearranging terms, one obtains Eq. (5). b is the desired deceleration of a car. – For practical purposes, one can use the average velocity $(v_{n-1} + v_n)/2$ for $\bar{v}$.

- The second rule (i.e. Eq. (6)) just states that the velocity is limited by the desired acceleration a, by the safe velocity v_{safe} as calculated above, and by the maximum velocity v_{max}.

- In the third term, noise ξ is added by randomly making the vehicle slower than so far calculated. ξ denotes a random variable between zero and one, η is a noise scaling factor.

- The fourth term simply denotes the forward movement.

Note that from the above derivation, the model is not automatically free of collisions, since Eq. (4) refers to continuous time, while the model (5) to (8) refers to discrete time. That the model indeed is free of collisions for time steps smaller than the reaction time τ is shown in Ref. [2].

This model shares some features with Gipps' model [24]. However, it is numerically more efficient, and it is better understood than the Gipps model, for which a thourough analysis as done in [2] is missing.

The model is stochastic, and it indeed follows the behavior lined out for stochastic models. However, the behavior of the model is not only understood for one particular setting of the parameters, but for arbitrary selections of the parameters a and b. Three different classes can be distinguished, which can be characterized as follows (see Figure 4, which also shows the corresponding typical fundamental diagrams of each class):

- Class "III": For large values of a and b, the model does not display any structure formation, i.e. the space-time plots look homogeneous. – The existence of this class is known for this model thanks to the thorough work in Ref. [2].

- Class "II": By decreasing a, but keeping large values of b, a regime is entered where the space-time plots show a complicated pattern of small jams within larger jams, which is basically fractal. This is the class where the stochastic cellular automaton model described in Sec. 5.3 can be found.
- Class "I": When approaching moderate values of a and b, and large-enough density, the space-time plot basically contain one large jam, that grows deterministically. The corresponding fundamental diagram displays a bistable behaviour (reversed λ–shape). This is the class where the stochastic CA models with slow-to-start rules (Sec. 5.4) can be found. This class is the most similar to the patterns observed in reality.

Thus, classes I and II correspond to behavior that we have already described in Sec. 5. Class III has never been described in the CA context, possibly because it does not look very relevant for traffic. Nevertheless, knowledge about the structure of a large parameter space seems extremely helpful.

Although this model is arguably the most realistic of all models presented in this paper, it has an interesting flaw: When lowering η, the break-down phenomenon vanishes completely for some $\eta < \eta_c$, $\eta_c > 0$ [28]. For small $\eta \geq 0$, the model simply has a stable homogeneous state $v_i \approx v(\rho)\,\forall\, i$.

The bistability can be recovered if the acceleration law is changed. By using a slow-to-start rule, for example by replacing $v(t) + a$ with

$$\begin{cases} v(t) + a_1 & \text{if } v(t) < v_{crit} \\ v(t) + a_2 & \text{else} \end{cases}$$

where $a_1 < a_2$, the fundamental diagram becomes bistable again.

The reason for this lies in the fact that the noise becomes biased for small velocities since we do not allow negative velocities. Therefore, the recipe that makes a fundamental diagram bistable is simply the slow–to–start rule mentioned earlier. The lesson of this is, though, really important: Starting from a physically motivated relation, such as Eq. (4), can result in a realistic model. But this is not necessarily the pathway to obtain the *simplest possible* model that generates certain mechanisms of realistic traffic.

7 Summary

It has been shown that different models of traffic flow display similar mechanisms that lead to traffic flow break-down or recovery after break-down, respectively. It can be assumed that the mechanisms are mathematically identical. The main difference is in the break-down mechanism between stochastic and deterministic models, as in the stochastic models a *strictly* bi-stable region does not exist: One of the branches of the density-flow relation becomes metastable or even unstable in the limit of very large system sizes. It can be concluded that the mechanism of traffic flow break-down in different models of traffic flow is well understood. Furthermore, the mechanisms that lead to recovery (or not) from break-down are understood as well.

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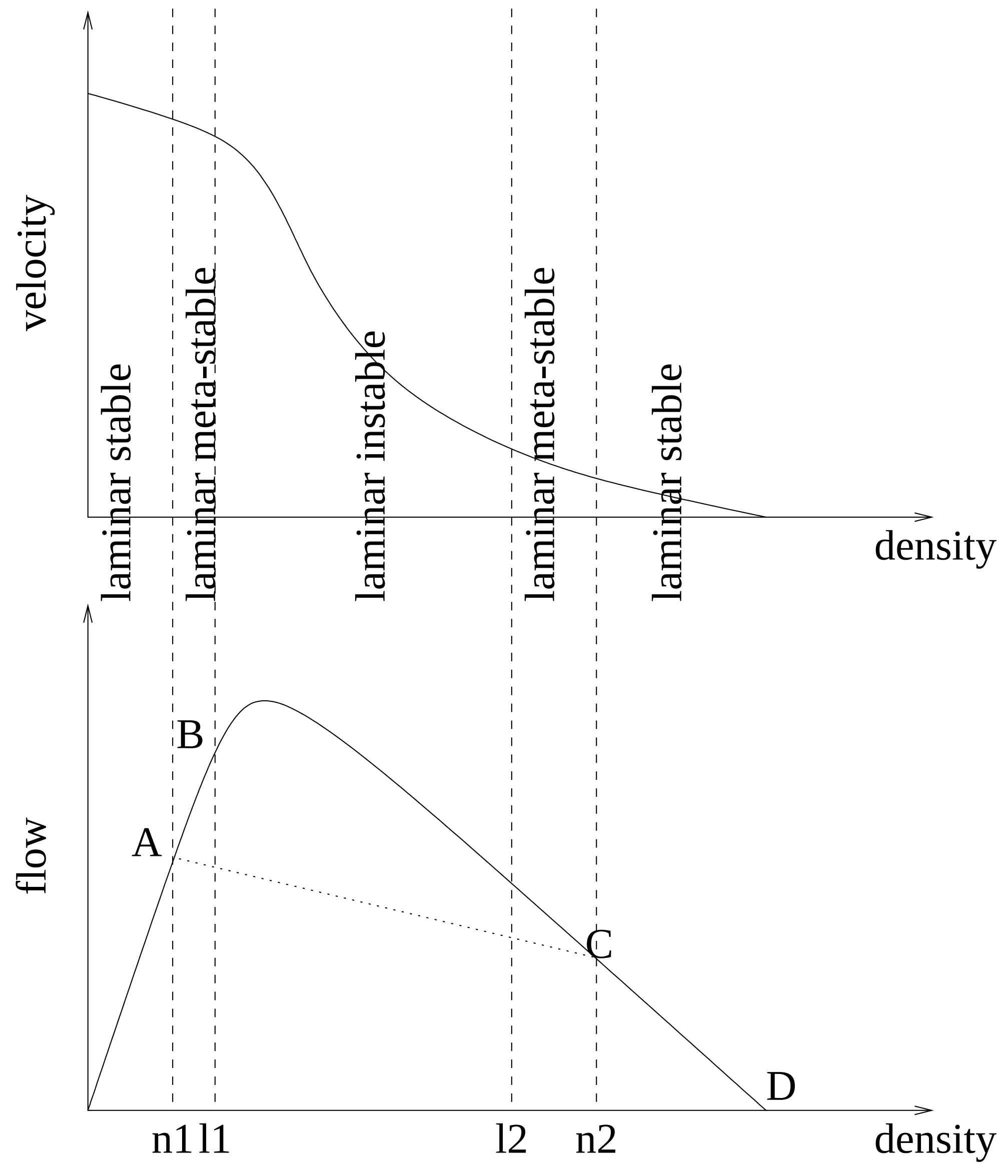

Figure 1: Stylized fundamental diagrams. The top diagram shows a possible “driving rule”, i.e. how desired speed may be related to density ρ, where ρ is the inverse of the front-bumper-to-front-bumper space headway. The full line in the bottom diagram shows the flow relation that would follow if laminar traffic were always stable. “n1”, “n2”, “l1”, “l2” denote ρ_{n1}, ρ_{n2}, ρ_{l1}, and ρ_{l2}. Between ρ_{l1} and ρ_{l2}, homogeneous traffic is linearly instable, that is, any smallest disturbance leads break-down and jam formation. ρ_{l1} and ρ_{l2} are the densities that one obtains from a linear stability analysis. Laminar traffic between ρ_{n1} and ρ_{l1} and again between ρ_{l2} and ρ_{n2} is stable in linear stability analysis but unstable for large enough fluctuations. Laminar traffic outside these densities is stable against *any* fluctuation. – In consequence, homogeneous traffic can occur up to point B. But after a large enough disturbance, it breaks down and is afterwards composed of parts operating at point C and parts operating at point A. Time averages of flow vs. density will measure arbitrary combinations between the two regimes, leading data points lying on the line connecting A and C.

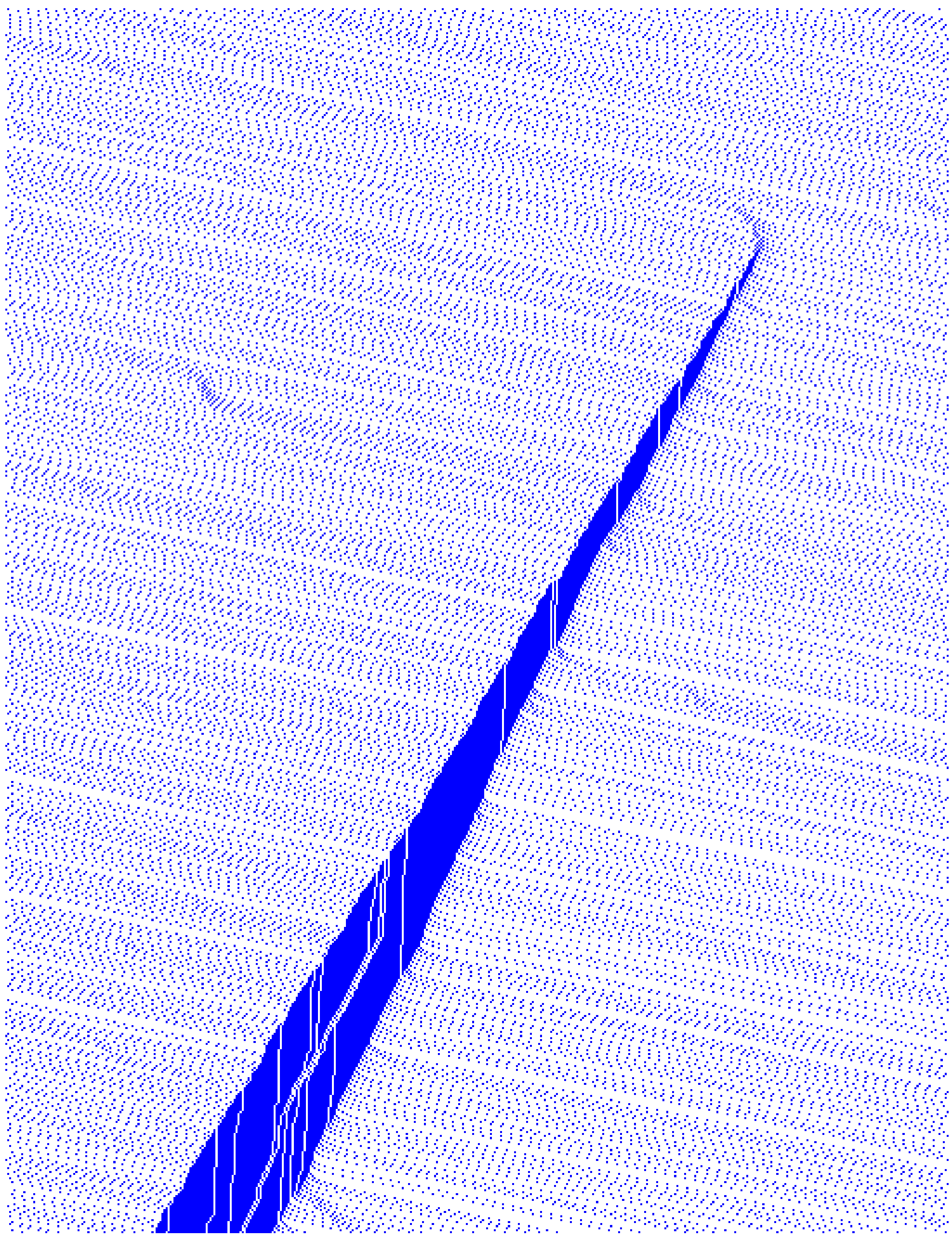

Figure 2: Space-time plot of a spontaneous traffic jam. Lines show consecutive time steps of configurations on a simulated 1-lane road; each pixel in a line is a different car.

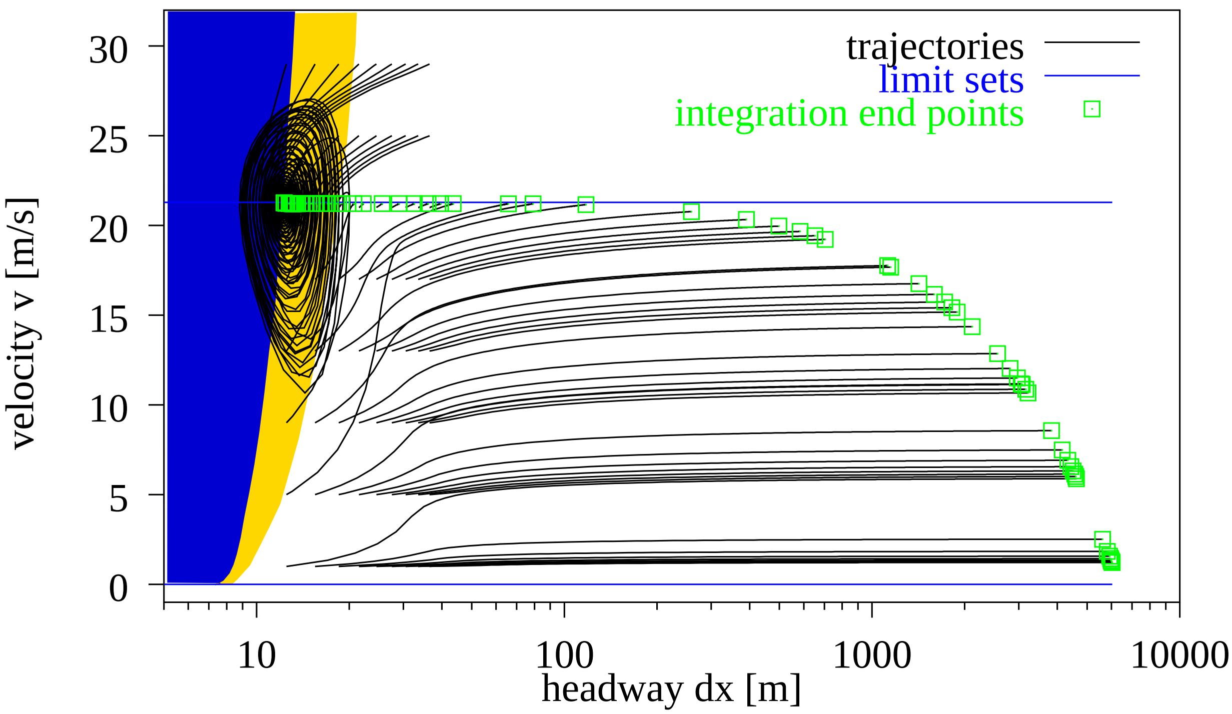

Figure 3: Phase portrait of eq. (2), with $\alpha = \tau = l = 1$ and $m = 1.7$. The velocity of the leading car is 22 m/s, plotted is Δx and v. Note that this picture is a two-dimensional projection of the infinite dimensional phase space of this model. Shown are the limit sets except the limit set $\Delta x = \infty$, and the end point of the various integrations. The initial conditions for any starting point in the $(\Delta x, v)$-plane is of course a function in the time interval $[-\tau, 0]$, where $(\Delta x(t) = g_{init}, v(t) = v_{init})\,\forall t \in [-\tau, 0]$ has been choosen. The unshaded region is the area in the phase plane where the fixed points $\Delta v = 0$ are a stable solution to eq. (2), while the region shaded in light grey has oscillatory damped solutions. The dark shaded region is the region where $\Delta v = 0$ is unstable.

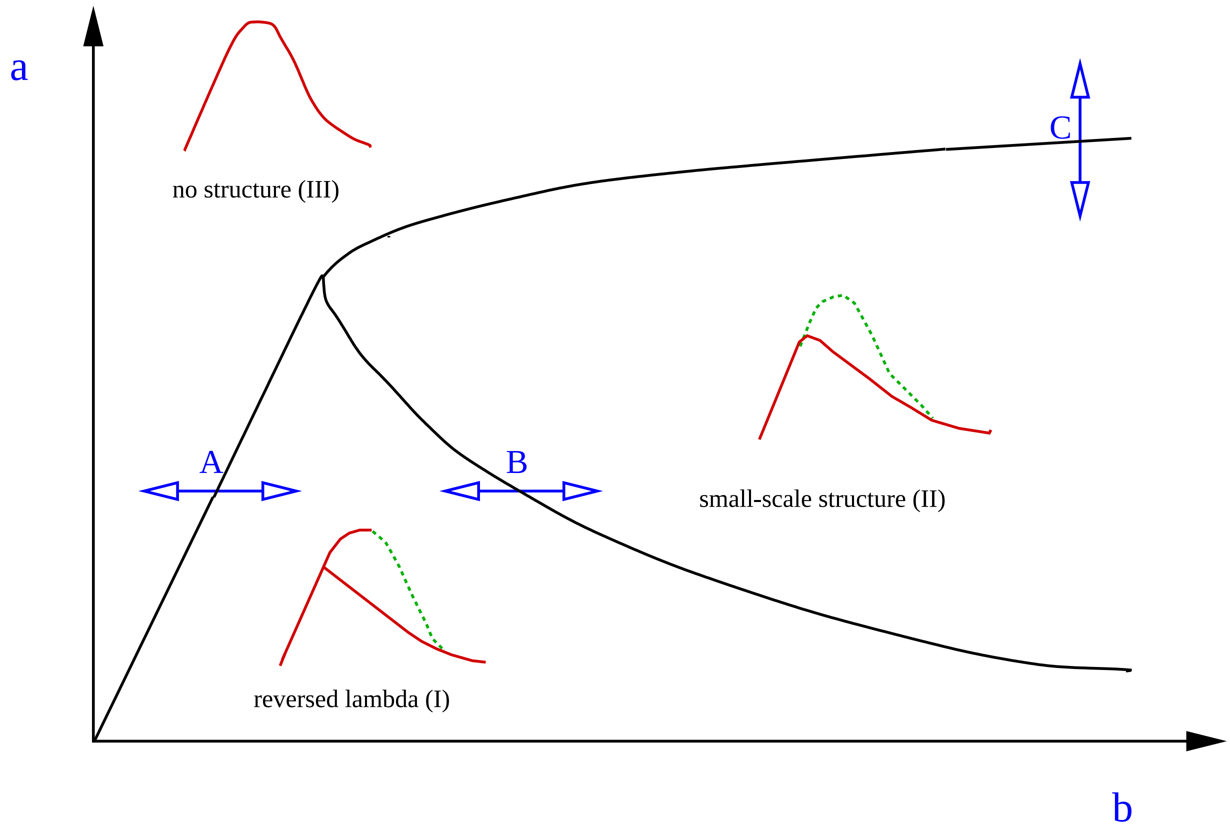

Figure 4: Phase diagram for the model family eq. (5 - 8), drawn in the acceleration deceleration (a, b) plane. The three model classes and their transition lines as described in the text are shown, together with their respective fundamental diagram. Realistic behaviour is found in the reversed lambda region, i.e. for moderate values of a and b.

```
.5.....3...00001.2..3...3...2..3...4....1.01.1.
5.....3...00001.2..3...3...2..3...4....1.01.1.2
.....3...00001.2..3...3...2..3...4....1.01.1.2.
....3...00001.2..3...3...2..3...4....1.01.1.2..
...3...00001.2..3...3...2..3...4....1.01.1.2..3
..3...00001.2..3...3...2..3...4....1.01.1.2..3.
.3...00001.2..3...3...2..3...4....1.01.1.2..3..
3...00001.2..3...3...2..3...4....1.01.1.2..3...
...00001.2..3...3...2..3...4....1.01.1.2..3...4
..00001.2..3...3...2..3...4....1.01.1.2..3...4.
```

Figure 5: Sequence of configurations of CA-184. Lines show configurations of a segment of road in second-by-second time steps; traffic is from left to right. Integer numbers denote the integer velocities that the particles/vehicles are about to execute. For example, a vehicle with speed "3" will move three sites (dots) forward. Via this mechanism, one can follow the movement of vehicles from left to right. – The backwards moving structures are just the kinematic waves according to the Lighthill-Whitham theory.