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Procedia Computer Science 109C (2017) 648–655

Procedia Computer Science

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The 8th International Conference on Ambient Systems, Networks and Technologies (ANT 2017)

The structure of user equilibria: Dynamic coevolutionary simulations vs. cyclically expanded networks

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Abstract

A variety of approaches exist that model traffic time-dependently. While all approaches have their advantages and disadvantages but have to find a balance between modeling traffic as realistic as possible and being still manageable in combinational terms. While transport simulations are efficient in evaluating user equilibria in large scale scenarios, their potential to be used for optimization is limited. On the other hand, analytical formulations like models based on cyclically time-expanded networks can be used to optimize traffic flow, but are not suitable for large scale scenarios. By optimizing the network structure in a mathematical model and evaluating its effect in a more realistic transport simulation, two models can benefit from each other. Detailed knowledge about model properties and differences in traffic flow behavior help to understand results and potential difficulties of such a model combination. In this paper, properties of two such models are compared regarding traffic flow modeling. It is shown that the set of user equilibria in both models and, therefore, the resulting route distributions can be structurally different.

Peer-review under responsibility of the Conference Program Chairs.

Keywords: transport modeling, transport simulation, cyclically time expanded networks, user behavior, user equilibria, system optimum

1. Introduction

In times were congestion levels are growing in many urban areas, there is a need to improve and refine transportation networks. Traffic models provide assistance with predicting traffic patterns and designing and evaluating traffic policies. A variety of modeling approaches exist. All of them have to make compromises between capturing the reality as good as possible and keeping the model complexity at a manageable level. Because of their simplicity, static flow models are widely used to optimize traffic management schemes like tolls, traffic signal plans, or other network adaptions. These models' theory is well established, e.g. in terms of the effect of selfish users on the system welfare.[1] Despite their time independence, static flow models can be used to model traffic of specific, fixed points in time where traffic flow can be assumed to be constant for a while, e.g. during rush hours.

In reality, however, traffic is *not* time-independent and travel times and demand change over time. There are approaches to translate static flow models into more realistic ones, capture time dependency but keep some of the

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Peer-review under responsibility of the Conference Program Chairs.
10.1016/j.procs.2017.05.371

properties to benefit from its simplicity. One idea is to expand the network over time by creating copies of every node and link per time step[2] (see section 2.1 for a detailed explanation of time-expansion). With this, flow travels over time in a static network. Time-expansion only works for constant, i.e. flow-independent link travel times. Otherwise, the properties of links in the time-expanded network would depend on route decisions of travelers. Constant link travel times seem to give realistic results in urban areas, where links are short, speed limits exist, and platoons of vehicles drive with a similar speed as single vehicles. Congestion occurs while waiting at signals or crossings and is modeled by waiting links at nodes. Route travel times then arise as the sum of constant link travel times and waiting times, which renders them non-constant again.[3] Hence, time-expanded models can capture dynamic flows with constant link travel times in a static network and at least some results on static flows are transferable.[4] A major disadvantage of time expansion is that the size of the network increases immensely compared to the size of the original network. Thus, applying optimization algorithms from static flow theory directly to time-expanded networks is no suitable approach in general. Still, it is possible to construct other algorithms using properties of time-expanded networks.[4]

An approach to handle the size of time-expanded networks is to expand the network only for a fixed, short time interval and cyclically combine the interval boundaries. This results in a manageable network size, but limited time dependency. Like in static flow models only stationary demand patterns are representable. At least, demand repeats in each cycle and does not have to be constant all the time.

In contrast to time-expanded networks where link travel times have to be constant, there are also approaches for flows over time with flow-dependent transit times. These lead to more realistic results, but also to mathematical difficulties.[5] Due to the lack of well-defined analytical models for this kind of flows, few results are known for them.

Another approach omits the analytical part and instead uses simulation tools. Transport simulation may capture a lot of the complex, realistic behavior of traffic flows like time-dependent demand and travel times, spill back to upstream parts of the network, and a more detailed user behavior that includes not only route, but also time and mode choice. This is done by an iterative approach that simulates agents traveling through the network and performing their daily activities. The daily plans of agents are then evaluated and some agents are allowed to re-plan their day until the iterations reach a stable state, i.e. no agent wants to change their plan anymore. Hence, transport simulation tools find user equilibria for complex systems where not all relations are known in terms of closed mathematical formulations. They result as fixed points of the iterative routing and assignment process.[6,7,8] On the other hand, simulation tools miss the optimization potential because of the complex system they capture.

Knowing the properties of the different models, one can try to find a combination of the different approaches which benefits from the advantages of the models while overcoming their specific weaknesses: While transport simulations are efficient in evaluating user equilibria in large scale scenarios, their potential to be used for optimization is limited. On the other hand, analytical formulations like models based on cyclically time-expanded networks can be used to optimize traffic flow, but are not suitable for large scale and highly time-dependent scenarios. By optimizing the network structure in the mathematical model and evaluating its effect in the more realistic transport simulation, both models can benefit from each other. Detailed knowledge about model properties and differences in traffic flow behavior helps to understand results and potential difficulties of such a model combination.

This paper compares two of the discussed approaches to model traffic in a time-dependent way: A cyclically time-expanded network model and a dynamic coevolutionary transport simulation. For the time-expanded model an approach by Köhler and Strehler at BTU Cottbus, which was developed for fixed-time traffic signal optimization, is considered.[3] On the other side, the transport simulation MATSim is used.[9] Both models have already been coupled to optimize fixed-time traffic signal plans in a real world scenario. For this, the scenario is provided by the transport simulation and converted into a cyclically time-expanded network. The static model then approximates optimal fixed-time signal plans for all signalized intersections by solving a mixed integer program (MIP) with the high performance solver CPLEX. These optimized signal plans are returned to the transport simulation to evaluate travel time effects in a more realistic model. Initial results have been presented by Grether[10].

The structure of this paper is the following: The two models are introduced in the next section and compared in section 2.3 regarding their model properties. Resulting flow patterns, i.e. user behavior of both models are compared in section 3. Conclusions are drawn in Section 4.

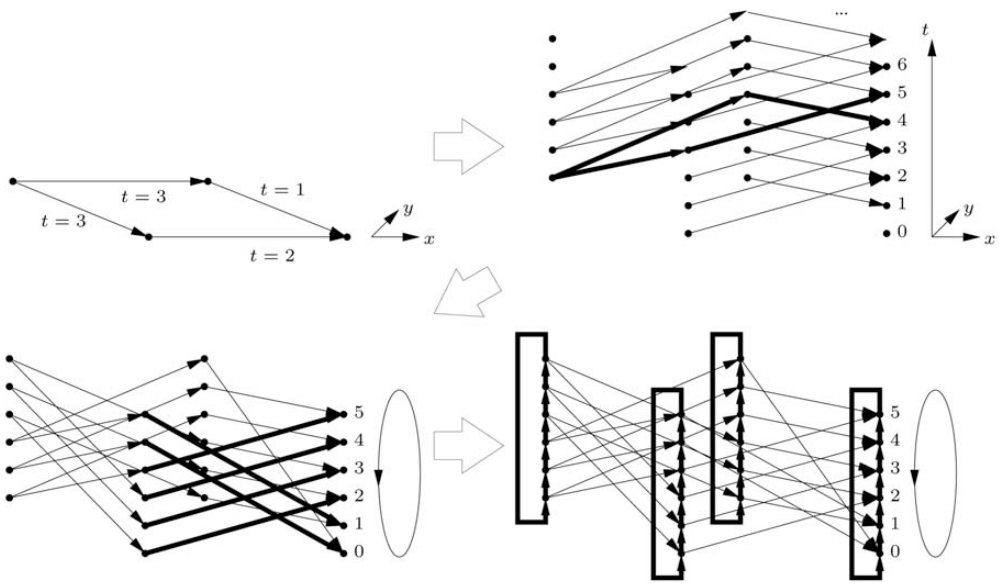

Fig. 1. Cyclically time-expanded network with waiting links: Expand the network over time (top right); cyclically combine it (bottom left); add waiting links (bottom right). [3]

2. Model properties

Both approaches studied in this paper – the cyclically time-expanded network model by Köhler and Strehler and the transport simulation MATSim – model traffic in a time-dependent way. In this section, the models are introduced and relevant model properties are compared in detail.

2.1. The cyclically time-expanded network model

The model of Köhler and Strehler was developed to optimize traffic signal coordination and traffic assignment simultaneously in an urban road network. [3] It is based on a time-expanded network, which uses the periodicity of traffic signals to limit the time horizon and, therefore, restrict computation time. In the following, it is called *the cyclically time-expanded network model*.

Time-expanded networks can model traffic in a time-dependent manner. Figure 1 illustrates time-expansion: Consider a static network with constant link travel times like in the upper left part of figure 1 with four nodes and four links. Choose a time step size (e.g. one second, as in figure 1) and create a (vertically) copy of each node for each time step. For every link in the static network, connect copies of the origin and destination node in the expanded network according to the constant travel time in the static network. This step can be seen in the upper right part of figure 1. When flow travels on one link in a time-expanded network, it automatically reaches the next node by the copy of the correct time step. In such a network, link flow values may differ for different time steps. Also demand values do not have to be stationary anymore in contrast to static networks. But the network size increases significantly. Therefore, the considered model limits the time horizon by taking advantage of the periodicity of traffic signals: The network is only expanded for a time interval of the size of the signal cycle time. Links are then added according to travel times modulo the number of time steps. This step is visualized in the lower left part of figure 1. A disadvantage of this cyclical concatenation is that some time dependency gets lost: Demand and link flow pattern have to be cyclically repeated. As a last step, waiting links, which allow flow particles to wait in front of intersections, are added to the network. This may be necessary because of link capacities that restrict inflow values per time step. Waiting links in the cyclically time-expanded network are illustrated in the lower right part of figure 1.

Although link travel times are constant, resulting route travel times of travelers behave not constant for increasing demand values. This is because more and more waiting arcs have to be used – i.e. waiting times increase – with increasing demand. (See Köhler and Strehler [3] for a detailed study on travel times in this model.)

The multi-commodity traffic assignment problem in the cyclically time-expanded network is analytically formulated together with signal coordination constraints in a corresponding mixed integer program (MIP). The program has a linear objective function that minimizes total travel time and, therefore, results in a *system optimum* (SO). To solve the mixed integer program, the high performance solver CPLEX is used. CPLEX iteratively calculates primal and dual bounds to search for a good solution of the problem and, on the other hand, to prove its optimality by closing the gap between primal and dual solutions. In some scenarios with many conflicting streams and high demand values, the gap can not be closed at all by CPLEX.[3]

2.2. The dynamic coevolutionary transport simulation

In contrast to static models and models that are based on time-expanded networks, a dynamic transport simulation is not based on a closed mathematical formulation that minimizes an objective, but simulates single agents traveling through a network and selfishly minimizing their travel time. It is, therefore, able to simulate traffic demand and travel times that changes over time.

The multi-agent transport simulation MATSim[9] considered in this paper belongs to the class of dynamic coevolutionary transport simulations. It is based on a network with free-speed travel times and link lengths, i.e. constant free-flow travel times for links like in the time-expanded network. Outflow rates are restricted by link flow capacities. Additionally, links have storage capacities that restrict the number of vehicles that can queue on a link. MATSim links are modeled as queues: Vehicles that enter a link queue up and are finally allowed to exit the link when they have reached the front of the queue, their free flow link travel time is reached, and flow capacity of the current link and storage capacity of the next link are not exceeded. Agents, i.e. synthetic travelers, depart and arrive on arbitrary links at arbitrary times which is modeled by daily plans. Plans contain a schedule of activities, including times and locations, along with the travel modes. Routes are also assigned to plans.

MATSim iterates between two major components: At first, the demand is simulated on the physical network (called *mobsim* for mobility simulation in figure 2), i.e. every agent executes its selected plan. Travel times and, therefore, activity durations of the executed activity travel pattern differ from times and durations in the plan because of congestion. The second major component of the iterative process is the mental simulation: Agents evaluate their decisions (called *scoring* in figure 2) and eventually replan them (called *replanning* in figure 2). Plans are evaluated based on their performance, which is quantified by a score. Scores sum up as utilities for all activity participations and times spent in traffic. Agents are allowed to select a plan for the next iteration. A certain percentage of agents is chosen to generate a new plan by modifying an existing plan. Possible modification strategies are e.g. route, time, or mode choice. The remaining agents select one of their already existing plans through probabilistic selection by a multinomial logit model, where the selection probability of a plan is related to its score.

Over the iterations, agents intend to maximize their score. The iterative process is repeated until agent scores do not vary, i.e. agents do not want to change their strategy, anymore. If scores converge, the process leads to a (stochastic) *user equilibrium* (UE), i.e. no user may improve his score by unilaterally changing his strategy. MATSim's learning dynamics, i.e. finding user equilibria as fixed points of the iterative routing and assignment process, are very similar to the approach presented by Cominetti. He proved that such learning dynamics converge almost surely towards a stationary state, which can be characterized as a user equilibrium.[8]

To be able to compare both models, this paper confines on situations where maximizing individual scores is similar to minimizing individual travel times in MATSim. Furthermore, stochasticity that comes from the probability of plan selection is significantly reduced.

2.3. Comparing both models

The models described in the two previous subsections – the cyclically time-expanded network model and the dynamic coevolutionary transport simulation – both aim to predict traffic flow with more or less aspects of time dependence, link travel times that are flow-independent and the possibility of waiting in front of intersections. Besides these aspects, there are many differences in both models (see Table 1). While comparing solution structures of both models and before coupling them to construct and evaluate traffic policies for real-world scenarios, it is important to analyze the properties and behavior of both models, along with underlying assumptions and their capabilities. This aims at better understanding coherences and consequences of coupling both models.

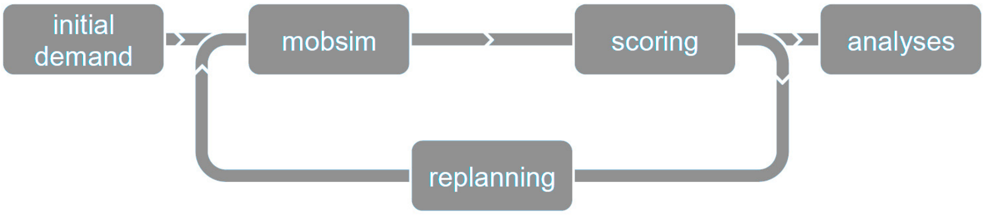

Fig. 2. The iterative transport simulation MATSim.

The most important difference is related to the way of capturing time dependency: In the cyclically time-expanded network, demand has to be the same for every cycle. In contrast to static flow models, at least variations in demand can be modeled, which cause variations in waiting times because of congestion and, therefore, variations in travel times. These variations are the same every cycle. In the dynamic coevolutionary transport simulation, demand and therefore travel times may vary arbitrarily over the day.

Although link travel times in both models are constant and flow dependency arises by link capacities that cause waiting in front of intersections, one can observe a small, but important difference here: Because of the cyclic structure, waiting times for flow particles, i.e. synthetic travelers, are bounded by one cycle length in the analytical model. The reason is that after one cycle the following copy of the synthetic traveler, belonging to the next cycle, arrives. If the former traveler was still there, delay would be accumulated, which is not manageable in a cyclical model. In MATSim, vehicles can wait unboundedly if necessary. Daily plans with long waiting times are then scored poorly and users try to find better ones.

When modeling vehicles that queue on a link, one usually distinguishes between models with point queues and spatial queues. In a model with point queues, queuing vehicles do not occupy space and, thus, do not influence whether following vehicles may enter the link or not. The cyclically time-expanded network model introduced in section 2.1 belongs to the class of point queue models: The waiting links with unbounded capacity exactly represent point queues at nodes; following vehicles are not influenced by the number of vehicles on the waiting link belonging to the link. In a model with spatial queues, queuing vehicles occupy space and can, therefore, spill back to upstream links. The transport simulation MATSim belongs to this class of models: The interplay between link length and vehicle size results in a maximum number of vehicles that fit on a link. If this number is exceeded, following vehicles have to wait on upstream links before they are allowed to enter the observed link.

Both models respect link flow capacities in terms of the maximum flow that can be processed by a link in a time period. The cyclically time-expanded network model uses link capacities as an entering restriction for links while the dynamic coevolutionary transport simulation uses them as exiting restrictions. So, in the analytical model congestion builds up upstream of the bottleneck whereas in MATSim it occurs on the actual bottleneck and directly upstream of it because of the spill back effects discussed in the last paragraph.[1]

Another difference is related to the way traffic flow is handled physically: In the time-expanded network model, no vehicles are considered. Flow values may split up in arbitrarily small flow particles to different routes. To ensure flow preservation in every node, the sum of all entering flow values (from links and from the node itself as an origin) has to be equal to the sum of all exiting flow values (to links and to the node itself as destination). In MATSim, this condition is not required. Flow is the sum of all individual vehicles, which cannot disappear or split, but have to travel from their start to their end link.

In the time-expanded network queuing takes place at the waiting links and not on the links that cover a distance. Because of this, passing of flow particles becomes possible: Following flow particles from another origin-destination pair may directly cross the intersection, while previous flow particles use the waiting link. In the agent-based simulation MATSim, links are directly represented by queues and therefore fulfill the first-in-first-out (FIFO) property. So, no passing is possible in the simulation.

[1] One could also model inflow capacities in MATSim. In urban networks with short link lengths this will not give a structurally different solution. This is due to the fact that MATSim simulates spill back effects. It would, by contrast, result in a totally different traffic flow pattern if one switches between inflow and outflow capacities in a scenario with long link lengths. [11] The reason is that spill back effects are disabled with long link lengths.

Table 1. Overview on similarities and differences of both models. The left column belongs to the cyclically time-expanded network model (named *KS model* here), the right one to the dynamic coevolutionary transport simulation MATSim.

	KS model	MATSim
Demand	stationary	time-dependent
Link travel times	constant	constant
Waiting times	bounded (cycle time)	unbounded
Queues	point (waiting links)	spatial
Capacities	inflow	outflow
Physical model	flow preservation	mass preservation
Priority	passing possible	FIFO
Optimum	SO = UE	SO ≤ UE

A final difference in between the two models consists in the applied objective function. The cyclically time-expanded network model determines a route distribution that minimizes total travel time, which means that it finds the system optimum. In the model, this route distribution constitutes a user equilibrium (See section 3.1. Other user equilibria with higher travel times may exist, however. The dynamic coevolutionary transport simulation, on the other hand, iteratively maximizes individual scores and results in a (stochastic) user equilibrium which does not necessarily minimize total travel time, even if score only includes travel times. (See the following section for a detailed discussion on this difference.)

Route distributions resulting from the models are mainly influenced by the fact that one model minimizes total travel time and the other individual travel time. But all other model properties discussed before also influence the solution properties. The next section uses knowledge about all properties and directly compares the outcome, i.e. the traffic pattern the models result in.

3. System optima and user equilibria

Selfish user behavior does not necessarily lead to a minimal total travel time, i.e. the system optimum, neither in theory nor in reality. For static flow models it has been shown that the difference of total travel time in user equilibrium and system optimum may become arbitrarily large even in small networks.[12] Also in a transport simulation, total travel times of user equilibrium and system optimum may differ unboundedly.[11]. A typical goal is to improve the travel time of user equilibria. As it is not possible to directly force the users to follow the system optimal routes, one could try to indirectly force them by modifying parts of the network or designing traffic policies. By optimizing the network structure in the mathematical model and evaluating its effect in the more realistic transport simulation, this can be modeled. But what impact does it have that optimization assumes a system-optimal route distribution whereas simulation evaluates it with the assumption of selfish users? This section compares properties of system optima and user equilibria in cyclically time-expanded network models and dynamic coevolutionary transport simulations.

3.1. System optima and user equilibria in the cyclically time-expanded network model

The cyclically time-expanded network model of Strehler and Köhler assumes travelers to follow the system-optimal routes. Due to constant link travel times, the system optimum is always a user equilibrium in their model in the sense that no user can improve their travel time by changing their route. To prove this, consider a system-optimal route distribution and an arbitrary infinitesimal user that considers changing their route. Link travel times are constant and, therefore, independent of the amount of flow particles using it. Changing the user's route would therefore not change any link travel times and by association not affect other users. It will also not improve travel time of the specific user; otherwise, total travel time would also improve, which is a contradiction to the system optimality of the considered distribution. Hence, no user can decrease their travel time by changing their route.[3]

Although the system optimum in this model is a user equilibrium, not every user equilibrium is system-optimal. In contrast to static models without capacity restrictions, multiple user equilibria may exist in capacitated networks. Users can occupy links and restrict the route choice for other users. The travel time of the worst user equilibrium can

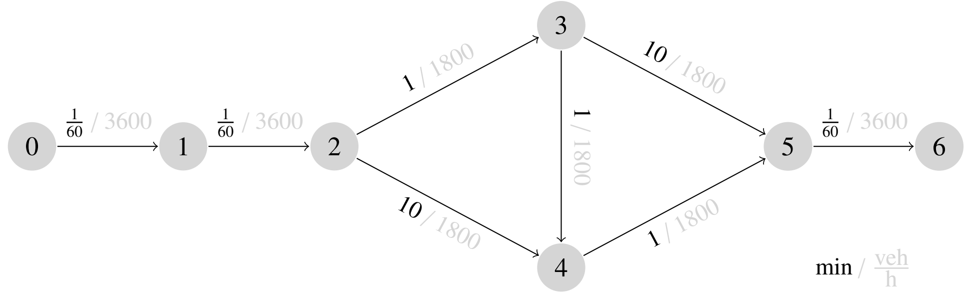

Fig. 3. Example network for section 3.2 with constant free flow travel times and capacities at the links.

even be arbitrarily different to that of the system optimum,[13,14] which corresponds to the best user equilibrium due to constant link travel times.

3.2. Comparing user equilibria of both models

As proved in section 3.1, the solution found by the cyclically time-expanded network model is a user equilibrium: Considering the set of feasible, alternative routes in the model, no user can improve their travel time. Nevertheless, the dynamic coevolutionary transport simulation which also results in a user equilibrium does not necessarily find the same traffic pattern as one can see in the following example.

Consider the network shown in figure 3. Free speed travel times and flow capacities are given for each link. 3600 vehicle per hour travel from the left to the right. There are three possible routes: The upper, the middle, and the lower route. If all users chose the middle route, which is the fastest with free speed travel times, the network could only handle 1800 vehicles per hour. In contrast, the network could handle the double amount of 3600 vehicle per hour if the users uniformly distribute to the two outer routes.

The scenario is based on Braess's paradox which states that removing links from a network may improve total travel time in user equilibria.[2] It presents the difference of both models regarding capacity handling: The cyclically time-expanded network model finds the system optimum, which is the uniform distribution to the outer routes. From the perspective of this model, the solution also represents a user equilibrium because it is the only feasible solution. Due to the cyclical repetition, waiting is not allowed for more than one cycle time (see section 2.3). As a consequence, flow particles can not switch to saturated, i.e. occupied links. In the dynamic coevolutionary transport simulation, agents can switch to saturated links, queue there and delay following agents. Because the middle route is the shortest from a selfish perspective, the simulation results in a distribution where all agents use the middle route, queue there, and only 1800 vehicles arrive per hour. From the perspective of the simulation, this situation represents a user equilibrium, i.e. no user may unilaterally improve their travel time, although total travel time is higher than in the system optimal outer distribution.

Hence, both models result in different traffic patterns, each constitute a user equilibrium for this scenario. As discussed in section 3.1, multiple user equilibria may exist in capacitated networks. The cyclically time-expanded network model finds the best equilibrium. Does this explain the differences and does the simulation just find a worse user equilibrium?

Instead, the set of user equilibria in both models differ structurally: The route distribution found by the transport simulation does not constitute a feasible solution of the cyclically time-expanded network model and, hence, no user equilibrium because capacities are strict and time horizon is limited due to the cyclical expansion – if users switch to the middle route, no route will be left for other users to choose because waiting times are bounded. The uniform distribution to the outer routes is the only feasible solution in the cyclically time-expanded network model. On the other hand, the solution that the cyclically time-expanded network model finds does not constitute a user equilibrium

[2] Braess's paradox was originally developed for static flow models without capacities,[15,16] but can also be observed in static flow models with capacities and also in time-dependent models. For a study how this paradox behaves in the dynamic coevolutionary transport simulation MATSim, see[11].

in the coevolutionary transport simulation. Users may improve their travel time by unilaterally switching to the middle route, although total travel time gets higher.

Both user equilibria are therefore not only different equilibria in the set of multiple user equilibria that may exist in both models. Instead, the strategy set of possible route distributions and therefore the models set of user equilibria is structurally different. As one example is enough to falsify equality, this shows that both models may result in structurally different user equilibria in general.

4. Conclusion

In this study, two different approaches to model traffic in a time-dependent way have been considered: An analytical approach based on a cyclically time-expanded network that gives time-dependent aspects although it is based on static flow theory and a simulation approach based on a coevolutionary, iterative learning algorithm and a queue-based representation of the network. Properties of both models as well as user equlibria that result as route distributions of travelers have been compared. It has been shown that both models differ in many aspects regarding time-dependent traffic flow and, in particular, the sets of user equilibria in both models are structurally different. Travel time of user equilibria found by both models may differ arbitrarily.

Despite and because of all differences, both models can be combined to benefit from each other: The analytical model can benefit from the simulation as a tool that evaluates the optimized network design in a more realistic environment and with users that behave selfishly; the simulation can benefit from the analytical model as a tool that produces suggestions for traffic policies on one hand and a system-optimal traffic pattern for this situation to compare with. If resulting traffic pattern are similar, it confirms the positive effect of the optimized traffic policy on total travel time. Potential differences in traffic pattern can be used to identify weak points of the optimized control policy and stabilize it before proposing it for reality. It can be assumed that many of the discussed aspects probably also hold for combinations of other mathematical network models with agent-based transport simulations.

Acknowledgements.

The author wish to thank E. Köhler and M. Strehler for their assistance in comparing and combining their model with MATSim and all the discussions about model details. The authors also thank the Deutsche Forschungsgemeinschaft (DFG) for funding the project *Optimization and network wide analysis of traffic signal control*.

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